How I Learned to Stop Worrying and Love Definable Henselian Valuations

Simone Ramello$^{\text{he/him}}$

joint with Margarete Ketelsen (Münster) and Piotr Szewczyk (Dresden)

Institut für Mathematische Logik und Grundlagenforschung
University of Münster, Germany

21.09.2023

MM
Mathematics Münster
Cluster of Excellence

The road ahead

I am a $\underbrace{\text{model theorist}}_{\text{LOGIC}}$ of $\underbrace{\text{valued fields}}_{\text{ALGEBRA}}$.

Today, I will try to:

- Tell you what valued fields are.
- Tell you why model theorists care.
- Give you an idea of what results in this area look like.

Valuations and where to find them

- Many fields come endowed with an *absolute value*, i.e. a notion of "size". A real number r is *large* if its absolute value $|r|$ is.
- Without such a notion of size or distance, we can't even ask certain questions!
- Valuations are a generalization of absolute values which arise in many interesting situations.

Valuations and where to find them, cont'd

Definition 0.1

A valuation on a field K is a surjective map $v\colon K^\times \to \Gamma$, where $(\Gamma, +, \leq, 0)$ is an ordered abelian group, such that:

- $v(xy) = v(x) + v(y)$, *multiplying two elements sums their valuations*
- $v(x + y) \geq \min\{v(x), v(y)\}$. *all triangles are isosceles*

(Counter)intuition: an element $r \in K^\times$ is *large* if if its valuation $v(r) \in \Gamma$ is *small*, i.e. close to 0.
Along this intuition, we usually set $v(0) := \infty$.

My favourite example

Fix a prime number p.

- If $a \in \mathbb{Z} \setminus \{0\}$, then

$$v_p(a) := \max\{n \in \mathbb{N} \colon p^n \mid a\}.$$

  For example, $v_3(6560) = 0$. According to v_3, then, 6560 is "big". But $v_3(6561) = 8$, which is then "smaller" than 6560. If $a, b \in \mathbb{Z} \setminus \{0\}$ are coprime, then

$$v_p\left(\frac{a}{b}\right) := v_p(a) - v_p(b).$$

- This defines a valuation $v_p \colon \mathbb{Q} \setminus \{0\} \to \mathbb{Z}$, called *the p-adic valuation*. We can turn it into an ultrametric absolute value $|\cdot|_p$ on $\mathbb{Q}$ by $|x|_p := p^{-v_p(x)}$.
- If we *complete* the corresponding metric space, we obtain a (new) valued field called $\mathbb{Q}_p$, with its own absolute value (and hence valuation) $|\cdot|_p$. These are the *p-adic numbers*.

WHY YOU SHOULD LIKE THE p-ADICS

- $(\mathbb{Q}_p, v_p)$ is crucial for algebraic purposes. But I'm a logician (allegedly)!
- A valuation is "the same" as its valuation ring, i.e. the subring

$$\mathcal{O}_v = \{x \in K \mid v(x) \geq 0\}.$$

This is the part where I should tell you that ∞ is larger than all elements of Γ, and thus $0 \in \mathcal{O}_v$.

- In the case of $\mathbb{Q}_p$, this subring is called $\mathbb{Z}_p$ (guess why!). Julia Robinson pointed out something remarkable about $\mathbb{Z}_p$ (for $p \neq 2$):

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid \exists Y(Y^2 = 1 + px^2)\}.$$

There is a similar formula for $p = 2$.

- $\mathbb{Z}_p$ is given, as a subset of $\mathbb{Q}_p$, by a polynomial equation together with some quantifiers. We say that it is a *definable* set in the language of rings.

Logicians, assemble! cont'd

Big question: Is this common? When is some valuation ring definable in the language of rings?

The problem of henselianity

Not all valuations are created equal.

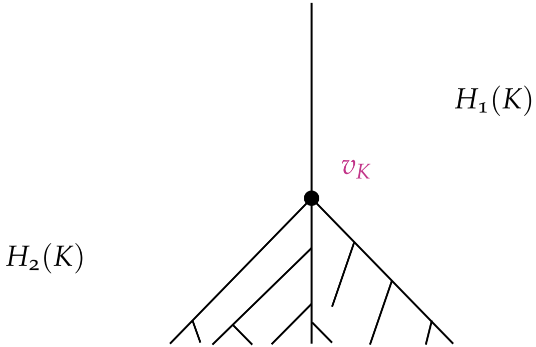
- Take a field K with a valuation v. I give you an algebraic extension L of K, e.g. $L = K(\alpha)$ where α is the root of some polynomial over K. Can you extend v to L?
  Yes, but often in several different ways.
- v is *henselian* if there is a **unique** way to extend v to any algebraic extension of K. A henselian valuation is a bit like a *fill the gaps* exercise in a textbook.
- v_p is henselian. *We will only care about henselian ones.*

The big question, take 2

Big question: how often is
an henselian valuation ring definable in the language of rings?

A canonical friend

- To any valued field (K, v) we can associate another "smaller" field, called the *residue field*,
$$Kv := \{x \in K \colon v(x) \geq 0\} \big/ \{x \in K \colon v(x) > 0\}.$$
Indeed, $\mathfrak{m}_v := \{x \in K \colon v(x) > 0\}$ is the unique maximal ideal of $\mathcal{O}_v = \{x \in K \mid v(x) \geq 0\}$.
- Henselian valuations on a given field K arrange themselves nicely according to whether their residue field is separably closed or not,
$$H_1(K) := \{v \colon Kv \neq Kv^{\mathrm{sep}}\} \text{ vs. } H_2(K) := \{v \colon Kv = Kv^{\mathrm{sep}}\}.$$
- $H_1(K)$ is linearly ordered by inclusion.
The "middle point" between $H_1(K)$ and $H_2(K)$ is the *canonical henselian valuation* v_K.



What we knew already

Theorem 1 (Jahnke, Koenigsmann, '17)

Let K be a non-separably closed henselian field such that Kv_K has characteristic 0. Then,

$$K \text{ admits a definable non-trivial henselian valuation} \iff \begin{cases} Kv_K = Kv_K^{\mathrm{sep}}, & \text{or} \\ Kv_K \text{ is not t-henselian}, & \text{or} \\ \exists L \succeq Kv_K \text{ with } v_L L \text{ divisible}, & \text{or} \\ v_K K \text{ is not divisible}. \end{cases}$$

What we proved

Theorem 2 (Ketelsen, R., Szewczyk, '23+)

Let K be a non-separably closed henselian field such that K is perfect of characteristic p. Then,

$$K \text{ admits a definable non-trivial henselian valuation} \iff \begin{cases} Kv_K = Kv_K^{\mathrm{sep}}, & \text{or} \\ Kv_K \text{ is not } t\text{-henselian}, & \text{or} \\ \exists L \succeq Kv_K \text{ with } v_L L \text{ divisible}, & \text{or} \\ v_K K \text{ is not divisible}, & \text{or} \\ (K, v_K) \text{ is not defectless}, & \text{or} \\ \exists L \succeq Kv_K \text{ with } (L, v_L) \text{ not defectless}. & \end{cases}$$

The gist of it

$$\underbrace{\exists \text{ definable (non-trivial) henselian valuation}}_{\text{Logic question}} \iff \underbrace{\text{Conditions on a certain canonical valuation}}_{\text{(Almost) algebra answer}}$$