Model theory of endomorphisms of valued fields

Oberseminar Logik

Motivation. for each $p>0$, consider $(\mathbb{F}_p(t)^{sep}, v_t, x\mapsto x^p) =: K_p$. The ultrapower $(K, v, \phi) := \prod_{p\to\mathcal{U}} (\mathbb{F}_p(t)^{sep}, v_t, x\mapsto x^p)$ is an algebraically closed valued field on which a non-surjective endomorphism acts "like an infinite Frobenius". ϕ is an endomorphism of (K,v) that is ω-increasing: for every $x\in K$, if $v(x)>0$, then for every $n\geq 0$, $v(\phi(x)) > n\cdot v(x)$. The theory of (K,v,ϕ) is well understood thanks to work of Dor & Halevi.

Q: what about general endomorphisms of valued fields?

(K,v,σ) "valued difference field"

I. Notation & Examples

All fields have characteristic zero.

If (K,v) is a valued field, then

- Γ_K value group,
- $\mathcal{O}_K = \{x\in K \mid v(x)\geq 0\} \supseteq \mathfrak{m}_K = \{x\in K \mid v(x)>0\}$

unique max. ideal

- $\overline{K} = \mathcal{O}_K/\mathfrak{m}_K$ residue field, $\mathrm{res}: \mathcal{O}_K \twoheadrightarrow \overline{K}$.

Ex. k char. zero, Γ o.a.g.

$$k((\Gamma)) = \{ \sum_{\gamma\in\Gamma} a_\gamma t^\gamma : \{a_\gamma\}_\gamma \subseteq k, \{\gamma : a_\gamma\neq 0\}\subseteq\Gamma \text{ w.o.}\}$$

are the power series,

$$v\Big(\sum_{\gamma\in\Gamma} a_\gamma t^\gamma\Big) = \min\{\gamma : a_\gamma \neq 0\}$$

is a valuation with $\Gamma_k = \Gamma$ & $\bar{K} = k$.

Def. consider the 3-sorted language $\mathcal{L}_{DF}$:

$$\begin{tikzcd} (K,+,\cdot,0,1,-) \arrow[r, bend right, "v"'] \arrow[rr, bend right, "ac"'] & (\Gamma_K,+,0,\leq,\infty) & (\bar{K},+,\cdot,0,1,-) \end{tikzcd}$$

where we interpret ac as an angular component, i.e. a group hom. $ac: K^\times \to \bar{K}^\times$ s.t., for all $u \in \mathcal{O}_K^\times$,

$$ac(u) = res(u).$$

Ex. in $k((\Gamma))$,

$$\sum_{\gamma\in\Gamma} a_\gamma t^\gamma \longmapsto a_{\min\{\gamma : a_\gamma\neq 0\}}$$

is an angular component. In general, the existence of an ac is not obvious.

Theorem. (Pas) Let $Hen_{0,0}$ be the $\mathcal{L}_{DF}$-theory of valued fields (K,v) s.t.

- $char(K) = char(\bar{K}) = 0$
- (K,v) is henselian, i.e. $\forall f \in \mathcal{O}_K[x]$ & $a \in \mathcal{O}_K$, if $v(f(a)) > 0$ and $v(f'(a)) = 0$, then $\exists b \in \mathcal{O}_K$ s.t. $f(b) = 0$ and $v(a-b) > 0$.

Then, modulo $Hen_{0,0}$, every $\mathcal{L}_{DP}$-formula is equivalent to one where quantifiers only range over Γ_K & $\bar{K}$.

↳ relative quantifier elimination

II. Enter ~~endo~~ auto morphisms

Given $\sigma \in \mathrm{End}(K,v)$, one gets $\bar\sigma \in \mathrm{End}(\bar K)$ & $\sigma_\Gamma \in \mathrm{End}(\Gamma_K)$.

Def. Consider the 3-sorted language $\mathcal{L}^\sigma_{DP}$, extending $\mathcal{L}_{DP}$,

$$\begin{tikzcd} (K,+,\cdot,0,1,-,\sigma) \arrow[r, bend right, "v"'] \arrow[rr, bend right, "ac"'] & (\Gamma_K,+,0,\leq,\infty) & (\bar K,+,\cdot,0,1,-,\bar\sigma) \end{tikzcd}$$

where ac is now interpreted as "σ-equivariant".

Theorem. (Durhan-Onay)

Let $\mathrm{Hen}^\sigma_{0,0}$ be the $\mathcal{L}^\sigma_{DF}$-theory of valued difference fields (K,v,σ) s.t.

- $\mathrm{char}(K) = \mathrm{char}(\bar K) = 0$, σ surjective,
- (K,v,σ) is σ-henselian.

Then, modulo $\mathrm{Hen}^\sigma_{0,0}$, every $\mathcal{L}^\sigma_{DP}$-formula is equivalent to one where quantifiers only range over Γ_K & $\bar K$.

↳ relative quantifier elimination

* A classical consequence: for $\square \in \{\equiv, \preceq\}$, then if $(K,v,\sigma), (L,v,\sigma)$ are models of $\mathrm{Hen}^\sigma_{0,0}$ & if $\square = \preceq$ then $(K,v,\sigma) \leq (L,v,\sigma)$,

$$(K,v,\sigma) \,\square\, (L,v,\sigma) \iff \big((\bar K,\bar\sigma) \,\bar\square\, (\bar L,\sigma) \;\&\; (\Gamma_K,\sigma_\Gamma) \,\square\, (\Gamma_L,\sigma_\Gamma)\big).$$

AK/E-style principles

Also, (K,v,σ) NTP_2 $\iff$ $(\bar K,\bar\sigma)$ & (Γ_K,σ_Γ) NTP_2 (Chernikov-Hils).

III. Beyond surjectivity

Def. Consider the 3-sorted language $\mathcal{L}_{DP}^{\sigma,\lambda}$, extending $\mathcal{L}_{DP}^{\sigma}$,

$$(K, +, \cdot, 0, 1, -, \sigma, (\lambda_n^i)_{n\geq 0}^{i\leq n}),\ (\bar{K}, +, \cdot, 0, 1, -, \bar{\sigma}),\ (\Gamma_K, +, 0, \leq, \infty, \sigma_\Gamma)$$

$$K \xrightarrow{ac} \bar{K}, \qquad K \xrightarrow{v} \Gamma_K$$

where we interpret λ_n^i as the following $(n+1)$-ary function:

$$\lambda_n^i(x_1, \dots x_n, y) = \begin{cases} z & \text{if } x_1, \dots x_n \text{ are } \sigma(K)\text{-lin. ind.}, \\ & y \in \langle x_1, \dots x_n \rangle_{\sigma(K)}, \text{ and} \\ & \sigma(z) \text{ is the } i\text{th coordinate} \\ & \text{of } y \text{ in the basis } \{x_1, \dots x_n\}. \\ 0 & \text{otherwise.} \end{cases}$$

Theorem. (R.)

Let $\mathrm{Hen}_{0,0}^{\sigma,\lambda}$ be the $\mathcal{L}_{DP}^{\sigma,\lambda}$-theory of valued difference fields (K, v, σ) s.t.

- $\mathrm{char}(K) = \mathrm{char}(\bar{K}) = 0$, $\bar{\sigma}$ & σ_Γ surjective,
- (K, v, σ) is weakly σ-henselian,
- $\sigma(K) \subseteq K$ is rel. alg. closed.

Then, modulo $\mathrm{Hen}_{0,0}^{\sigma,\lambda}$, every $\mathcal{L}_{DP}^{\sigma,\lambda}$-formula is equivalent to one where quantifiers only range over Γ_K & $\bar{K}$.

↳ relative quantifier elimination

What is the point of λ? [Dor-Halevi]

Rem. K field of pos. char. p, then $K \subseteq L$ is separable iff $K \underset{K^p}{\overset{l.d.}{\perp\!\!\!\perp}} L^p$.

If $K \subseteq L$ is algebraic, this the same as $\forall a \in L, \exists f \in K[x]$ s.t. $f(a) = 0$ & $f'(a) \neq 0$. Separability "preserves" imperfection:

K perfect $\Leftrightarrow$ $K^p = K$ $\Leftrightarrow$ all ext^{n}s are separable.

The least separable extension is $K^{perf} = \bigcup_{n \geq 0} K^{-p^n}$.

<u>Def.</u> $(K,\sigma) \leq (L,\sigma)$ is transf. separable if $K \overset{\text{l.d.}}{\underset{\sigma(K)}{\downarrow}} \sigma(L)$.

Again, σ surjective $\Leftrightarrow$ $\sigma(K) = K$ $\Leftrightarrow$ all ext^{n}s are t-sep.

The least t-sep. ext^n is $K^{inv} := \bigcup_{n \geq 0} \sigma^{-n}(K)$.

<u>Rem.</u> $(K,\sigma) \leq (L,\sigma)$ is t-sep. iff K is closed under the λ-functions. Moreover, if the extension is σ-alg. & $\sigma(K)$ is rel. alg. closed in K, then t-sep. $\Leftrightarrow$ $\forall a \in L \ \exists f \in K[x]_\sigma$ s.t. $f(a) = 0$ & $f'(a) \neq 0$.

"FE" in Dor-Halevi & Chatzidakis-Hrushovski

derivation of $f(X, \sigma(X), \ldots \sigma^n(X))$ along X,
eg $f = \sigma(X) - a$, then $f' \equiv 0$

Proving relative QE. (sketch)

$$\begin{tikzcd}
 & (L,v,\sigma) \arrow[dl, phantom] \\
(K,v,\sigma) \arrow[ur, hook, dashed] & \\
(A,v,\sigma) \arrow[u, no head] \arrow[uur, no head] &
\end{tikzcd}
\quad \& \quad
\begin{tikzcd}
 & (\bar{L},\bar{\sigma}) \\
(\bar{K},\bar{\sigma}) \arrow[ur, hook] & \\
(\bar{A},\bar{\sigma}) \arrow[u, no head] \arrow[uur, no head] &
\end{tikzcd}
,\quad
\begin{tikzcd}
 & (\Gamma_L,\sigma_\Gamma) \\
(\Gamma_K,\sigma_\Gamma) \arrow[ur, hook] & \\
(\Gamma_A,\sigma_\Gamma) \arrow[u, no head] \arrow[uur, no head] &
\end{tikzcd}$$

for simplicity, assume A subfield

want to produce ▨ lifting the other two.

<u>Two steps:</u> ① ensure $\Gamma_A = \Gamma_K$ & $\bar{A} = \bar{K}$, ie $(A,v) \leq (K,v)$ is immediate,

σ-henselianity

② "Kaplansky theory".

② is the tricky part:

Theorem. (R.)

Suppose (K, v, σ) is a valued diff. field s.t.

1. $\sigma(K) \subseteq K$ is rel. alg. closed,
2. $(\bar{K}, \bar{\sigma})$ is linearly closed.

$\forall \alpha_0, \dots \alpha_n \in \bar{K}$, at least one non-zero, $\exists z \in \bar{K}$ s.t. $0 = 1 + \alpha_0 z + \alpha_1 \bar{\sigma}(z) + \dots + \alpha_n \bar{\sigma}^n(z)$

Then $\exists!$ $(\tilde{K}, v, \sigma)$ immediate, transf. sep., σ-alg. over K s.t.

up to K-iso

- $\tilde{K}$ has proper imm. t. sep. σ-alg. ext^n,
- $\sigma(\tilde{K}) \subseteq \tilde{K}$ is rel. alg. closed.

Ramification. In $\aleph_0$-saturated models of $\mathrm{Hen}^{\sigma,\lambda}_{0,0}$, one has

$$\bigcap_{n \geq 0} \sigma^n(K) = K_{inv} \subsetneq K \subsetneq K^{inv} = \bigcup_{n \geq 0} \sigma^{-n}(K)$$

dense

So many results from the surj. world can be transferred.