⋆ Blockseminar SoSe2022 ⋆

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episode 1: the henselian menace

to the tune of "Let it be"

When I find myself in times of trouble
Henselian valuations come to me
Equicharacteristic zero
A-K-E

GOAL:

THE AKE PRINCIPLE

If (K,v) and (L,w) are valued fields, then

$$(K,v) \equiv (L,w) \iff k_K \equiv k_L \ \wedge\ \Gamma_K \equiv \Gamma_L .$$

Recall: $v : K^\times \twoheadrightarrow \Gamma_K$, ordered abelian group (oag), $v(0) := \infty > \Gamma_K$

$\rightsquigarrow$ $\mathcal{O}_K = \{ x \in K \mid v(x) \geq 0 \} \supseteq \mathfrak{m}_K = \{ x \in K \mid v(x) > 0 \}$

$\rightsquigarrow$ $k_K = \mathcal{O}_K / \mathfrak{m}_K$ – residue field.

– for us today it means

- char$(k_K) = 0$,
- (K,v) is henselian.

$P(x) \in \mathcal{O}_K[X]$, then simple roots of $\bar{P}(x) \in k_K[X]$ lift to roots of P in K.

Choice of language

$\mathcal{L}_0$ = three-sorted language

$$\underbrace{\mathbb{K}}_{\mathcal{L}_{ring}} \quad \underbrace{\Bbbk}_{\mathcal{L}_{ring}} \quad \underbrace{\Gamma}_{\mathcal{L}_{oag} \cup \{\infty\}}$$

$v : \mathbb{K} \longrightarrow \Gamma$ interpreted as the valuation

$\mathcal{L}_{ring} = \{+, -, \cdot, 0, 1\}$ $\mathcal{L}_{oag} = \{+, \leq, 0\}$

Angular components

A map $ac : K \to k_K$ s.that

1. $ac(x) = 0$ iff $x = 0$,
2. $ac : K^\times \to k_K^\times$ is a group morphism
3. if $\pi : \mathcal{O}_K \twoheadrightarrow k_K$ is the residue map, then

$$ac(x) = \pi(x)$$

for $x \in \mathcal{O}_K^\times$,

is called an angular component map.

Motto: angular component maps exist in saturated extensions.

Main theorem

Denote by $\mathcal{L}_{Pas}$ the three-sorted language obtained from $\mathcal{L}_0$ by adding a symbol $ac : \mathbb{K} \to \Bbbk$. Let T_0 be the $\mathcal{L}_{Pas}$-theory that says, for a model $(K, k_K, \Gamma_K, v_K, ac_K) \models T_0$,

1. (K, v_K) is a henselian valued field,
2. $\mathrm{char}(k_K) = 0$,
3. if we define $\pi_K(x) := \begin{cases} ac_K(x) & \text{if } v_K(x) = 0, \\ 0 & \text{otherwise} \end{cases}$, then $\pi_K : \mathcal{O}_K \to k_K$ is a surjective ring map with kernel $\mathfrak{m}_K$.

For fixed k of char. 0 and Γ oag, $T = T_0$ "$\cup$" $\mathrm{Th}_{Ring}(k)$ "$\cup$" $\mathrm{Th}_{oag}(\Gamma)$.

Main theorem

T eliminates the $\mathbb{K}$-quantifier in $\mathcal{L}_{Pas}$.

let $\Sigma = \{$ $\mathcal{L}_{Pas}$-formulae with no quantifiers over the variables of sort $\mathbb{K}\}$,
then any $\mathcal{L}_{Pas}$-formula is equivalent, modulo T, to a formula in Σ.

Lemma:

Let T be a $\mathcal{L}$-theory, and let Σ be a set of $\mathcal{L}$-formulae closed under Boolean combinations. Suppose that, for some $\kappa > |T|$, for any $M, N \models T$ κ-saturated, for any $A \subseteq M$, $B \subseteq N$ with $|A| < \kappa$, for any $f : A \xrightarrow{\sim} B$ isomorphism that preserves Σ, for any $a \in M \setminus A$, we may extend f to an iso $f' : A' \xrightarrow{\sim} B'$, $|A'| < \kappa$, that preserves Σ with $a \in A'$. Then any $\mathcal{L}$-formula is equivalent, modulo T, to one in Σ.

PROOF STRATEGY start with two $\aleph_1$-saturated models of T:

$$\begin{array}{cccc} & (K, k_K, \Gamma_K) & & (L, k_L, \Gamma_L) \\ & \rotatebox[origin=c]{90}{$\succcurlyeq$} & & \rotatebox[origin=c]{90}{$\succcurlyeq$} \\ \text{countable} & (A, k_A, \Gamma_A) & \xrightarrow[(f, f_r, f_v)]{\sim} & (B, k_B, \Gamma_B) \end{array}$$

Take $a \in K - A$, take $(C, k_C, \Gamma_C) \preccurlyeq (K, k_K, \Gamma_K)$ countable with $a \in C$, we look for a recipe to extend (f, f_r, f_v) to (C, k_C, Γ_C) while preserving Σ.

We interweave different steps.

We interweave different steps:

6 (C, k_C, Γ_C)

3' (A_1^h, k_C, Γ_C)

5 $(A_1, k_C, \Gamma_C = \nu(A_1^\times))$

4 $(A_0, k_C = \pi(\upsilon_{A_0}), \Gamma_C)$

3 (A^h, k_C, Γ_C)

(A, k_C, Γ_C)

2

(A, k_C, Γ_C)

1

(A, k_A, Γ_A)

All steps feature a way to extend (f_l, f_r, f_v) to the new structure

STEP 0: ('the one that everyone forgets')

$\mathcal{L}_{ring}$ does not contain the "inverse" map, so $A \subseteq K$ and $k_A \subseteq k_K$ are just subrings. However, we may canonically extend

$$(f, f_r, f_v)$$

to $(\mathrm{Frac}(A), \mathrm{Frac}(k_A), \Gamma_A)$.

+ $v_{\mathrm{Frac}(A)}\left(a/b\right) := v_A(a) - v_A(b)$

+ $ac_{\mathrm{Frac}(A)}\left(a/b\right) := ac_A(a)/ac_A(b)$

We interweave different steps:

$$(C, k_C, \Gamma_C)$$

6

$$(A_1^h, k_C, \Gamma_C)$$

3

$$(A_1, k_C, \Gamma_C = \nu(A_1^\times))$$

5

$$(A_0, k_C = \pi(\vartheta_{A_0}), \Gamma_C)$$

4

$$(A^h, k_C, \Gamma_C)$$

3

$$(A, k_C, \Gamma_C)$$

2

$$(A, k_C, \Gamma_C)$$

1

$$(A, k_A, \Gamma_A)$$

All steps feature a way to extend (f_l, f_r, f_v) to the new structure

STEP 1:

Extend (f, f_r, f_v) to (A, k_C, Γ_A).

- enumerate $k_C - k_A = (c_n)_{n \in \mathbb{N}}$.
- inductively: let $p(x) = tp(c_0 / k_A)$, consider

$q(y) \ni \varphi(y, f_r(\bar{c})) \iff \varphi(x, \bar{c}) \in p(x), \ \bar{c} \in k_A$

and by saturation let $b_0 \in L$, $b_0 \models q$.

- now f_r extends to

$$f_r' : k_A(c_0) \xrightarrow{\sim} k_B(b_0).$$

- repeat throughout $(c_n)_{n \in \mathbb{N}}$.

We obtain an $\mathcal{L}_{Pas}$-isomorphism (f, f_r', f_v) defined on (A, k_C, Γ_A).

STEP 1: (cont.'d)

The map (f, f'_r, f_v) preserves formulae in Σ:

in fact, it is enough to preserve formulae of the form

$$(\star) \quad \underbrace{\psi_0(x_0)}_{\substack{\text{quantifier-free} \\ \mathcal{L}_{ring},\ x_0 \in K}} \wedge \underbrace{\psi_1(ac(t_1(\bar{x})), y_1)}_{\substack{\mathcal{L}_{ring}\text{-formula}, \\ x_1, y_1 \in k,\ t_1 \text{ term} \\ \text{in } \mathcal{L}_{ring},\ \bar{x} \in K}} \wedge \underbrace{\psi_2(v(t_2(\bar{x})), y_2)}_{\substack{\mathcal{L}_{oag} \cup \{\infty\}\text{-formula}, \\ x_2, y_2 \in \Gamma,\ t_2 \text{ term} \\ \text{in } \mathcal{L}_{ring},\ \bar{x} \in K}}$$

and in particular f_r preserves $\psi_1(ac(t_1(\bar{x})), y_1)$, so we are done.

We interweave different steps:

6 (C, k_C, Γ_C)

3' (A_1^h, k_C, Γ_C)

5 $(A_1, k_C, \Gamma_C = \nu(A_1^\times))$

4 $(A_0, k_C = \pi(\nu_{A_0}), \Gamma_C)$

3 (A^h, k_C, Γ_C)

(A, k_C, Γ_C)

2

(A, k_C, Γ_C)

1

(A, k_A, Γ_A)

All steps feature a way to extend (f_i, f_r, f_v) to the new structure

STEP 2: (like Step 1, but for groups)

Extend (f, f_r, f_v) to (A, k_C, Γ_C).

As in Step 1, we enumerate $\Gamma_C - \Gamma_A$ and add one element at a time.

The new map (f, f_r, f_v') will preserve Σ because of $(*)$.

Now we find ourselves with (f, f_r, f_v) defined on (A, k_C, Γ_C). In particular, for any $\bar{a} \in K$ we have f_r defined on $ac(t_1(\bar{a}))$ and f_v defined on $v(t_2(\bar{a}))$. Therefore any $\mathcal{L}_{Pas}$-isomorphism will preserve Σ as long as it extends (f, f_r, f_v).

We interweave different steps:

⑥ (C, k_C, Γ_C)

③' (A_1^h, k_C, Γ_C)

⑤ $(A_1, k_C, \Gamma_C = \nu(A_1^\times))$

④ $(A_0, k_C = \pi(\vartheta_{A_0}), \Gamma_C)$

③ (A^h, k_C, Γ_C)

(A, k_C, Γ_C)

② (A, k_C, Γ_C)

① (A, k_A, Γ_A)

All steps feature a way to extend (f_ι, f_r, f_v) to the new structure

STEP 3:

Note that, as $(C, k_C, \Gamma_C) \preccurlyeq (K, k_K, \Gamma_K)$, (C, v_C) is henselian. Hence, by the universal property, $A^h \hookrightarrow C$ and, similarly, $B^h \hookrightarrow L$.

In fact one might check that

$$A^h = A^{alg} \cap C \quad , \quad B^h = B^{alg} \cap L$$

so that f extends to $f' : A^h \xrightarrow{\sim} B^h$.

Then (f', f_r, f_v) is an $\mathcal{L}_{Pas}$-isomorphism

$$(A^h, k_C, \Gamma_C) \longrightarrow (B^h, k_B, \Gamma_B).$$

As in Step 2, this automatically preserves Σ.

interlude

One might be tempted to think of $k_C = k_A$ as the residue field of (A, v_A), and of $\Gamma_C = \Gamma_A$ as the value group. However, a priori we might have

$$v_A(A^\times) \subsetneq \Gamma_C \qquad\qquad ac_A(\mathcal{O}_A) \subsetneq k_C$$

so we need to make v_A and ac_A surjective.

We interweave different steps:

⑥ (C, k_C, Γ_C)

③' (A_1^h, k_C, Γ_C)

⑤ $(A_1, k_C, \Gamma_C = \nu(A_1^\times))$

④ $(A_0, k_C = \pi(\vartheta_{A_0}), \Gamma_C)$

③ (A^h, k_C, Γ_C)

(A, k_C, Γ_C)

②

(A, k_C, Γ_C)

①

(A, k_A, Γ_A)

All steps feature a way to extend (f_l, f_r, f_v) to the new structure

STEP 4:

We want an intermediate $(A, k_C, \Gamma_C) \subseteq (D, k_C, \Gamma_C) \subseteq (C, k_C, \Gamma_C)$ such that $ac_D(\mathcal{O}_D) = k_C$. Denote by $\bar{k_A} = ac_A(\mathcal{O}_A)$.

- $\alpha \in k_C \setminus \bar{k_A}$ transcendental over $\bar{k_A}$. Then $f_r(\alpha)$ is t. over $\bar{k_B}$, so any $a \in C$ with $\pi_C(a) = \alpha$, $b \in L$ with $\pi_L(b) = f_r(\alpha)$ are t. over A and B respectively. The isomorphism

$$f' : A(a) \xrightarrow{\sim} B(b)$$

is an $\mathcal{L}$pas-isomorphism: e.g.

$$v_L\left(\sum f(c_i) b^i\right) = \min_i v_L(f(c_i)) = f_v(\min_i v_C(c_i))$$
$$= f_v\left(v_C\left(\sum c_i a^i\right)\right).$$

$\boxtimes$ $\alpha \in k_C \setminus k_{\bar{A}}$ algebraic: let $\bar{P}(t) \in k_{\bar{A}}[t]$ be the min. polyn. for some $P \in \mathcal{O}_A[t]$, $\deg(P) = \deg(\bar{P}) = N$. Then α is a simple root of $\bar{P}$ and hence it lifts to $a \in \mathcal{O}_C$. Similarly, $f_r(\alpha)$ is a simple root of $\overline{P^{(f)}}(t)$ which lifts to $b \in L$. Again, f extends to $f' : A(a) \xrightarrow{\sim} B(b)$ and we have e.g.

$$v_L\Big(\sum_{i<N} f(c_i) b^i\Big) = \min_{i<N} v_L(f(c_i)) = f_v\big(\min_{i<N} v_C(c_i)\big)$$

$$= f_v\Big(v_C\Big(\sum_{i<N} c_i a^i\Big)\Big),$$

so (f', f_r, f_v) is an $\mathcal{L}_{\mathrm{val}}$-isomorphism.

We interweave different steps:

6. (C, k_C, Γ_C)

3'. (A_1^h, k_C, Γ_C)

5. $(A_1, k_C, \Gamma_C = \nu(A_1^\times))$

4. $(A_0, k_C = \pi(\upsilon_{A_0}), \Gamma_C)$

3. (A^h, k_C, Γ_C)

(A, k_C, Γ_C)

2.

(A, k_C, Γ_C)

1.

(A, k_A, Γ_A)

All steps feature a way to extend (f_l, f_r, f_v) to the new structure

STEP 5:

Like step 4, we have a dichotomy: let $\Gamma_A^- = v_C(A^\times)$, then

- $\alpha \in \Gamma_C \setminus \Gamma_A^-$ has no torsion mod Γ_A^-. Then any $a \in C$ with $v_C(a) = \alpha$ is transcendental over A, and similarly any $b \in L$ with $v_L(b) = f_v(\alpha)$ is t. over B. Then f extends to

$$f' : A(a) \xrightarrow{\sim} B(b)$$

as before. Since we may choose $ac_C(a) = 1 = ac_L(b)$, (f', f_r, f_v) is an $\mathcal{L}_{pas}$-isomorphism.

there is $N>0$ s.that $N\alpha \in \Gamma_A$.

Then we can choose $a \in C$ with $a^N \in A$, $v_C(a) = \alpha$. Similarly, we can choose $b \in L$ with $v_L(b) = f_v(\alpha)$. We may, wlog, also assume $ac_L(b) = f_r(ac_C(a))$. Upon multiplying by $d \in L$ with $d^N = 1+u$, where $f(a^N) = c^N(1+u)$ and $v_L(u) > 0$, $\pi_L(d) = 1$, we get $f_v(\alpha) = v_L(b)$ but now $ac_L(b) = f_r(ac_C(a))$. Thus

$$f' : A(a) \xrightarrow{\sim} B(b)$$

is an $\mathcal{L}_{Pas}$-isomorphism.

We interweave different steps:

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5 $(A_1, k_C, \Gamma_C = \nu(A_1^\times))$

4 $(A_0, k_C = \pi(\vartheta_{A_0}), \Gamma_C)$

3 (A^h, k_C, Γ_C)

(A, k_C, Γ_C)

2

(A, k_C, Γ_C)

1

(A, k_A, Γ_A)

All steps feature a way to extend (f_l, f_r, f_v) to the new structure

We repeat Step 3 to move to an henselian valued field again.

Recall: henselian valued fields in char. 0 have no immediate algebraic extension.

We interweave different steps:

6 (C, k_C, Γ_C)

3' (A_1^h, k_C, Γ_C)

5 $(A_1, k_C, \Gamma_C = \nu(A_1^\times))$

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3 (A^h, k_C, Γ_C)

(A, k_C, Γ_C)

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(A, k_C, Γ_C)

1

(A, k_A, Γ_A)

All steps feature a way to extend (f_l, f_r, f_v) to the new structure

STEP 6:

$$(A, k_C, \Gamma_C) \subseteq (C, k_C, \Gamma_C)$$

is now an immediate valued field extension.

In particular, any $a \in C \setminus A$ is transcendental over A and, with a sprinkle of Kaplansky theory, we get that for any $P(t) \in A[t]$ there is $\delta \in \Delta(a/A) = \{ v_C(a, c) \mid c \in A \}$ s. that $v(P(t))$ is constant on $B_\delta(a) \cap A$.

Claim: there is $b \in L$ s. that $\forall c \in A$

$$v_L(b - f(c)) = f_v(v_C(a-c)).$$

Claim: there is $b \in L$ s. that $\forall c \in A$

$$v_L(b - f(c)) = f_v(v_C(a-c)).$$

Then, $v_L(p^{(f)}(t))$ is constant on $B_{f_v(\delta)}(f(a'))$, for $a' \in B_\delta(a) \cap A$.
In particular, $a \mapsto b$ extends to an isomorphism of valued fields

$$f': A(a) \xrightarrow{\sim} B(b)$$

which extends f and induces an $\mathcal{L}$pas- isomorphism.

PROOF OF THE CLAIM: take $\pi(x) = \{ v_L(x - f(c)) = f_v(v(a-c)) \mid c \in A \}$.
Realize it in L by $\aleph_1$-saturation.

Upon repeating Step 6,
we finish the proof!

AKE

As a corollary, let's prove that

$$k_K \equiv k_L \ \wedge\ \Gamma_K \equiv \Gamma_L \Rightarrow (K,v) \equiv (L,w).$$

($\Leftarrow$ is immediate!)

This is equivalent to proving that the $\mathcal{L}_0$-theory T_1, given by

1. (K,v) is henselian of equichar. 0,
2. $Th_{Ring}(k_K) = Th_{Ring}(k_L)$,
3. $Th_{oag \cup \{\infty\}}(\Gamma_K) = Th_{oag \cup \{\infty\}}(\Gamma_L)$,

is complete. Any two models of T_1 can be extended to models of T, by saturation!

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1. (K,v) is henselian of equichar. 0,
2. $\mathrm{Th}_{\mathrm{Ring}}(k_K) = \mathrm{Th}_{\mathrm{Ring}}(k_L)$,
3. $\mathrm{Th}_{\mathrm{oag}\cup\{\infty\}}(\Gamma_K) = \mathrm{Th}_{\mathrm{oag}\cup\{\infty\}}(\Gamma_L)$,

is complete. Any two models of T_1 can be extended to models of T, by saturation!

Now, if $(K, k_K, \Gamma_K), (L, k_L, \Gamma_L) \models T$ were not elementarily equivalent, this would be witnessed without K-quantifiers. However, $k_K \equiv k_L$ and $\Gamma_K \equiv \Gamma_L$. $\lightning$

thanks!