Towards a model theory of Frobenius lifts

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GOAL/HOPE: understand the model theory and algebra of unramified henselian valued fields, enriched with a lift of the Frobenius from the (imperfect) residue field.

In the perfect case, BMS prove an AKE-type principle for a class of structures of the form

$$(K, v, \sigma)$$

which includes the case $K = W(Kv)$, Kv perfect infinite of char. $p > 0$, and σ lifts $\bar{\sigma} : Kv \xrightarrow{\sim} Kv$ the Frobenius. A key example: $Kv = \mathbb{F}_p^{alg}$.

The result relies heavily on two steps: recall the coarsening diagram for henselian unramified valued fields,

$v_0 : K \to \Gamma/\Delta$, $\Delta \leq \Gamma$ the convex subgroup gen. by $v(p)$

$$K \twoheadrightarrow Kv_0 \twoheadrightarrow Kv$$

If you want to prove AKE in the pure case, starting out with

$$(K, v), \quad (L, w), \quad vK \equiv wL, \quad Kv \equiv Lw$$

under enough saturation you may assume $vK \cong wL$, $Kv \cong Lw$ and then it's a "lifting game":

$$\begin{tikzcd} (K, v_0) \arrow[r, two heads] \arrow[d, phantom, "\cong"] & Kv_0 \arrow[r, two heads] \arrow[d, dash, "2"] \arrow[l, phantom, "\textcircled{2}"] & Kv \arrow[d, dash, "2"] \arrow[l, phantom, "\textcircled{1}"] \\ (L, w_0) \arrow[r, two heads] & Lw_0 \arrow[r, two heads] & Lw \end{tikzcd}$$

① is achieved with Witt vectors (for perfect residue) and Cohen rings (for imperfect residue - Anscombe, Jahnke)

② is achieved through the classical $(0,0)$ AKE

In the enriched case, the situation is similar:

Kv, Lw perfect $\Rightarrow$ $Kv_0 \cong W(Kv)$, $Lw_0 \cong W(Lw)$

$$\begin{tikzcd}
(K, v_0, \sigma) \arrow[r, two heads] \arrow[d, phantom, "\equiv"] \arrow[dr, phantom, "②"] & (Kv_0, \sigma_0) \arrow[r, two heads] \arrow[d, phantom, "\wr"] \arrow[dr, phantom, "①"] & (Kv, \bar{\sigma}) \arrow[d, phantom, "\wr"] \\
(L, w_0, \tau) \arrow[r, two heads] & (Lw_0, \tau_0) \arrow[r, two heads] & (Lw, \bar{\tau})
\end{tikzcd}$$

① is the structure theory of Witt vectors

② requires a $(0,0)$ argument, that BMS give through σ-Kaplansky theory

~~What about imperfect Kv and Lw?~~

Both ① & ②, as they are, fail:

- Witt vectors aren't useful for imperfect residue,
- σ-Kaplansky for non-surjective maps is unclear.

Our hope: supplement ① with Cohen rings (Anscombe - Jahnke).

Let's focus on ①.

Obstacles:

- lifts of the Frobenius are not unique
- Cohen rings are not rigid (in other words, they are unique up to non-unique iso)

$\leadsto$ there is much more data to keep track of! (due to p-bases & representatives)

→ we focus on <u>one</u> type of lift. Let $C[k]$ Cohen over k.

Fix $\bar{\beta} \subseteq kv$ a p-basis and $s: \bar{\beta} \to C[k]$ a choice of representatives.

One can embed $C[k]$ inside $W[k^{perf}]$ using s and $\bar{\beta}$. There is a unique (!) lift of the Frobenius to $W[k^{perf}]$.

Let $\phi: C[k] \to C[k]$ be the restriction of the Frobenius to $C[k]$.

Call ϕ naive of type $(s, \bar{\beta})$. Eg., $\phi(s(b)) = s(b)^p$ $\forall b \in \bar{\beta}$.

(Being "of some type" is first-order in finite deg. of impf.)

of some type (⇒ also of some type)

$$\begin{tikzcd}
(K, v, \phi) \arrow[r, two heads] \arrow[d, phantom, "\cong"] & (kv_0, \bar{v}, \phi_0) \arrow[r, two heads] \arrow[d, phantom, "\cong"] \arrow[r, phantom, shift right=4, "(*)"] & (kv, \bar{\phi}) \arrow[d, phantom, "\cong"] \\
(L, w, \psi) \arrow[r, two heads] & (Lw_0, \bar{w}, \psi_0) \arrow[r, two heads] & (Lw, \bar{\psi})
\end{tikzcd}$$

Thm. Suppose $kv \cong Lw$, then $(kv_0, \bar{v}, \phi_0) \cong (Lw_0, \bar{w}, \psi_0)$. (*)