Why should you care about the model theory of valued fields

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The plan today:

- what is model theory (hopefully)
- the analogy between $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ (in algebra)
- what model theorist can say about this analogy

We will really move back-and-forth between the two first points, hopefully motivating the first through the second.

What is model theory?

The goal of model theory is tackling mathematical structures in a formal way, i.e. properties that can be expressed by a formula in a formal language.

Our guiding example: the language of rings,

$$\mathcal{L} = \{ \underbrace{\boxed{+, \cdot, -,}}_{\text{function symbols}} \underbrace{\boxed{0, 1}}_{\text{constant symbols}} \}$$

where $\mathcal{L}$-formulae are built in this way: start with

$$\underbrace{p(X_1, \dots X_n)}_{\text{polynomials}} = \underbrace{Q(Y_1, \dots Y_m)}_{\text{over } \mathbb{Z}}$$

and then recursively use connectives:

- negation ($\neg$) $\rightsquigarrow$ $\neg(p(x_1,\dots x_n)=0)$ means $p(x_1,\dots x_n)\neq 0$,
- conjunction ($\wedge$) & disjunction ($\vee$),
- implication ($\to$),
- quantifiers ($\forall,\exists$) $\rightsquigarrow$ $\forall x,y\,(x^2-y=0)$, $\exists z_1,\dots z_n\,(Q(\bar{z})\neq 0)$.

Some examples: $\forall x(x\neq 0 \to \exists y\,(x=y^2))$

$\exists x\,(x^2+1=0)$

Both of these examples are $\mathcal{L}$-sentences, i.e. all variables that occur are quantified:

$$\forall x\,(x\neq 0 \to \exists y\,(x=y^2)).$$

These are the formulae that make sense in a structure:

$$\exists y\,(x=y^2)$$

is this true? It depends on the choice of x!
Let's focus on sentences, then. We then need to formalize the idea of "being true": $\mathcal{L}$-structures.

$\mathcal{L}=\{+,\cdot,-,0,1\}$ $\rightsquigarrow$ $(\mathbb{R},+,\cdot,-,0,1)$
but also,
$(\mathbb{R},\cdot,+,-,0,1)$
$(\mathbb{R},\cdot,+,e^x,1,0)$
...

We give meaning to a sentence in an $\mathcal{L}$-structure via these choices of interpretations. We will use $\models$.

Some examples:

$\mathbb{R} \models \forall X (X \neq 0 \to \exists Y (X = Y^3))$

we mean $(\mathbb{R}, +, \cdot, -, 0, 1)$

$\mathbb{R} \not\models \exists X (X^2 + 1 = 0)$

"false"

$\mathbb{C} \models \exists X (X^2 + 1 = 0)$

we mean $(\mathbb{C}, +, \cdot, -, 0, 1)$

$\mathbb{R}_{exp} \models \forall X, Y (\exp(X+Y) = \exp(X) \cdot \exp(Y))$

in an expanded language $\mathcal{L} \cup \{\exp\}$

$\overline{\mathbb{R}} \models \forall X (0 \leq X \to \exists Y (X = Y^2))$

in an expanded language $\mathcal{L} \cup \{\leq\}$

Given a structure M in some language $\mathcal{L}'$,

$$\mathrm{Th}(M) = \{ \varphi \ \mathcal{L}'\text{-sentence} : M \models \varphi \}.$$

Rephrasing the goal: understand M in terms of $\mathrm{Th}(M)$. As an example, we will look at

$$\mathrm{Th}(\mathbb{Q}_p) \overset{?}{\longleftrightarrow} \mathrm{Th}(\mathbb{F}_p((t)))$$

both in the language $\mathcal{L}$.

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the analogy between $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$

	$\mathbb{Q}_p$		$\mathbb{F}_p((t))$
value group	$\mathbb{Z}$		$\mathbb{Z}$
residue field	$\mathbb{F}_p$		$\mathbb{F}_p$
characteristic	0	!	p

This last line tells us something: even for $\mathcal{L}$,

$$\mathrm{Th}(\mathbb{Q}_p) \neq \mathrm{Th}(\mathbb{F}_p((t))).$$

$$\not\ni \qquad \ni$$

$$\underbrace{1+\dots+1}_{p\text{-times}} = 0$$

However, they "look" very similar:

$$\mathbb{Q}_p = \left\{ \sum_{n\geq N}^{\infty} a_n p^n \;\middle|\; N\in\mathbb{Z},\ a_n\in\{0,1,\dots p-1\} \right\},$$

$$\mathbb{F}_p((t)) = \left\{ \sum_{n\geq N}^{\infty} a_n t^n \;\middle|\; a_n\in\mathbb{F}_p,\ N\in\mathbb{Z} \right\}.$$

Indeed, up to moving to some expansion,

Theorem. (Fontaine – Wintenberger)

$$\mathrm{Gal}(\mathbb{Q}_p(p^{1/p^\infty})) \simeq \mathrm{Gal}(\mathbb{F}_p((t))(t^{1/p^\infty})).$$

(This is really a consequence of $\mathrm{Gal}(K) \simeq \mathrm{Gal}(K^\flat)$).

⚠ We need to toss in all those p-th roots. Indeed,

$$\mathrm{Gal}(\mathbb{Q}_p) \not\simeq \mathrm{Gal}(\mathbb{F}_p((t))).$$

One can actually prove that $\mathrm{Gal}(K) \simeq \mathrm{Gal}(\mathbb{Q}_p)$ if and only if $\mathrm{Th}(\mathbb{Q}_p) = \mathrm{Th}(K)$.
as topological groups

The model-theoretical content of FW has been recently explored by Jahnke & Kartas, using a harder version of the next theorem.

What can model theory say?

THEOREM. (Ax-Kochen / Eršov, '60s)
Given a sentence φ in the language $\mathcal{L}$, there is a natural number $N = N(\varphi) \in \mathbb{N}$ such that, for all $p \geq N$,

$$\varphi \in \mathrm{Th}(\mathbb{Q}_p) \iff \varphi \in \mathrm{Th}(\mathbb{F}_p((t))).$$

In other words, the two theories are asymptotically the same ("$p \to \infty$ makes p transcendental").

Let's see a concrete example: C_i fields.
C_i-fields arise as generalizations of algebraically closed fields:

$C_i(d)$ – homogeneous polynomials of degree d in $N > d^i$ variables have non-trivial roots

Then C_i means $C_i(d)$ for all $d \in \mathbb{N}$.
For example, finite fields are C_1 (note that C_1 just asks for $N > d$). Complete discretely valued fields with algebraically closed residue field are C_1 (e.g. the fraction field of $W[\mathbb{F}_p^{alg}]$).

Artin. $\mathbb{Q}_p$ is C_2. $\rightsquigarrow$ false, by Terjanian. However,

COROLLARY. There exists $\bar{p} = \bar{p}(d)$ such that, for all $p \geq \bar{p}$, $\mathbb{Q}_p$ is $C_2(d)$.

This is a corollary of AKE because
- being $C_2(d)$ is a sentence ϕ_d,
- $\mathbb{F}_p((t))$ is C_2 (this was proved by Lang).

What is model theory really saying?

The AKE principle is really much deeper: what is true for a large class of valued fields is that

$$\mathrm{Th}(K, v) = \mathrm{Th}(L, w) \iff \mathrm{Th}(k) = \mathrm{Th}(\ell) \ \& \ \mathrm{Th}(\Gamma_K) = \mathrm{Th}(\Gamma_L)$$

(expanded language) residue fields value groups