Definable henselian valuations & independent defect

(joint work with M. Ketelsen and P. Szewczyk)

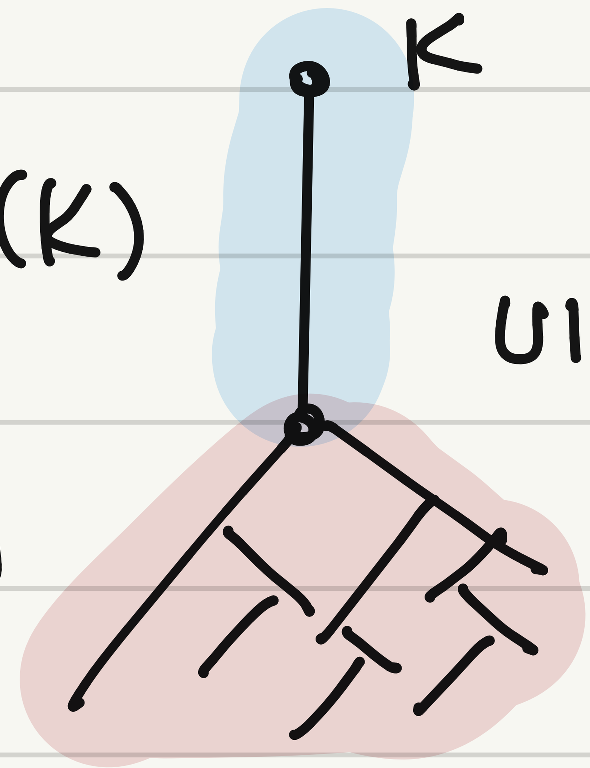
NOTATION.

K field, v valuation on K: Kv residue field, vK value gp
$\mathcal{O}_v$ valuation ring
$\mathfrak{m}_v$ maximal ideal

If K admits some non-trivial henselian valuation (i.e. K is henselian), then henselian valuations arrange themselves along "$\subseteq$" on valuation rings.

$$\{v \text{ on } K \mid Kv \neq Kv^{sep}\} = H_1(K)$$

$$\{v \text{ on } K \mid Kv = Kv^{sep}\} = H_2(K)$$



If $H_2(K) = \emptyset$, then define $\mathcal{O}_{v_K}$ to be the smallest one (the finest one) in $H_1(K)$. Otherwise, define it to be the biggest one (the coarsest one) in $H_2(K)$.

We call v_K the canonical henselian valuation on K. We will say that a valuation v is definable if $\mathcal{O}_v \subset K$ is $\mathcal{L}_{ring}$-definable (possibly with parameters).

MOTIVATION.

Theorem (Jahnke-Koenigsmann for equich. 0; Ketelsen-R.-Szewczyk for pos. res. char.)
$K \neq K^{sep}$ henselian & perfect; if $\mathrm{char}(Kv_K) = p > 0$ and $\mathrm{char}(K) = 0$, then assume also $\mathcal{O}_{v_K}/p\mathcal{O}_{v_K}$ is semiperfect. Then:

K admits a definable non-trivial henselian valuation

$\Longleftrightarrow$

(1) $Kv_K = Kv_K^{sep}$, or
(2) Kv_K is not t-henselian, or
(3) there is $L \succeq Kv_K$ henselian, $v_L L$ not divisible, or
(4) $v_K K$ not divisible, or
(5) (K, v_K) not defectless, or
(6) there is $L \succeq Kv_K$ henselian, (L, v_L) not defectless.

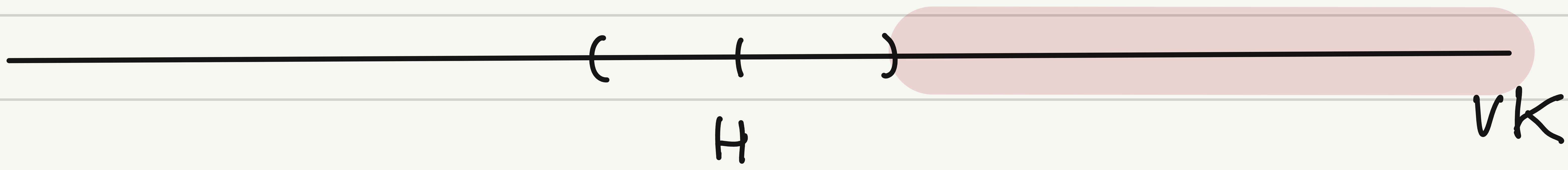
Today: I want to explain how (5) implies existence of a definable non-trivial henselian valuation.
If time permits, a few words on (6).

Note that in equich. 0, (5) and (6) are void.

A TOY EXAMPLE.

Suppose you are given a $\mathcal{L}_{ring}$-definable $D \subset K$ s.t. there is some convex subgroup $H < vK$ with

$$v(D) = \{\alpha \in vK \mid \alpha > H\}.$$



We would like to define the coarsening of v that corresponds to vK/H. Ideally, we would like to say – if v_H is the coarsening – that

$$v_H(x) > 0 \iff \cancel{x \in D}.$$

$\exists u, d \, (d \in D \;\&\; v_H(u) = 0 \;\&\; x = ud)$.

But this is not true! Instead ...

This is an "implicit definition" in the sense of Beth's definability.

Theorem (Beth def. - special case)
$\mathcal{L}' = \mathcal{L} \cup \{P(\bar{x})\}$
T some $\mathcal{L}'$-theory
Then TFAE:
(a) for all $(M, P_1), (M, P_2) \models T$, where M is some $\mathcal{L}$-structure, $P_1 = P_2$,
(b) there is $\varphi(\bar{x})$ in $\mathcal{L}$ such that
$$T \vdash \forall \bar{x} \, (P(\bar{x}) \leftrightarrow \varphi(\bar{x})).$$

Proposition. Suppose $v \in H_1(K)$ and $D \subset K$ is a definable set s.that, for some convex $H < vK$,
$$v(D) = \{\alpha \in vK \mid \alpha > H\}.$$
Then the coarsening v_H corresponding to H is definable.

PROOF. $\mathcal{L} = \mathcal{L}_{ring}$, $\mathcal{L}' = \mathcal{L}_{ring} \cup \{P(x)\}$,
$T = Th_{\mathcal{L}'}(K, \mathcal{O}_{v_H})$.
Take $(L, \mathcal{O}_1), (L, \mathcal{O}_2) \models T$. Then $\mathcal{O}_1, \mathcal{O}_2$ are henselian valuation rings in $H_1(L)$ and both satisfy, if $D = \psi(K)$, $i = 1, 2$,
$$x \notin \mathcal{O}_i \iff \exists u \in \mathcal{O}_i^\times \, \exists d \, (\psi(d) \wedge x = u \cdot \tfrac{1}{d}),$$
since this is true of $\mathcal{O}_{v_H}$.
Now, as $\mathcal{O}_1, \mathcal{O}_2$ are in H_1, they are comparable, e.g. $\mathcal{O}_1 \subset \mathcal{O}_2$.
Suppose $\mathcal{O}_1 \subsetneq \mathcal{O}_2$: if $x \in \mathcal{O}_2 \setminus \mathcal{O}_1$, then there are $u \in \mathcal{O}_1^\times$ and $d \in \psi(L)$ s.t. $x = u \frac{1}{d}$. But now $u \in \mathcal{O}_1^\times \subset \mathcal{O}_2^\times$, so $x \notin \mathcal{O}_2$, a contradiction.
Hence $\mathcal{O}_1 = \mathcal{O}_2$ & we apply Beth's def. $\square$

Idea: use defect to find such a D (in some finite ext^n).

DEFECT.

(K,v) henselian valued field with $p = \mathrm{char}(Kv)$ if >0
& $p=1$ otherwise

$K \subset L$ finite field extension, with unique extension of v (which we also call v)

$$\Rightarrow \quad [L : K] = p^d \, [Lv : Kv] \, (vL : vK),$$

for some $d \geq 0$. (fundamental inequality)

Definition. $(K,v) \subset (L,v)$ is defectless if $p^d = 1$.
(K,v) is defectless if all of its finite extensions are.

E.g. if $\mathrm{char}(Kv)=0$, then (K,v) is always defectless.

(*) for the rest of this talk, $\mathrm{char}(Kv) = p > 0$
$= \mathrm{char}(K)$.

If $(K,v) \subset (L,v)$ is Galois defect of degree p, then fix σ s.t. $\mathrm{Gal}(L|K) = \langle \sigma \rangle$ and write

$$\Sigma_L = \left\{ v\left(\frac{\sigma(f)-f}{f} \right) : f \in L^\times \right\}.$$

Facts. (Kuhlmann-Rzepka)
Σ_L is independent of σ and $\Sigma_L \subset vK$ is a final segment.

Definition. $(K,v) \subset (L,v)$ as above has independent defect if there is a convex $H < vK$ s.t.

vK/H has no minimum positive element &
$$\Sigma_L = \{\alpha \in vK \mid \alpha > H\}.$$
We say it has dependent defect otherwise.

Idea dependent defect comes from purely inseparable immediate extensions outside the completion.
In particular, if K is perfect, there are no dependent defect extensions (Proposition 1.14, Kuhlmann-Rzepka).

<u>Theorem.</u> Suppose (K,v) is a perfect henselian valued field with $v \in H_1(K)$, (K,v) not defectless. Then some non-trivial coarsening of v is $\mathcal{L}_{ring}$-definable.

PROOF.

Claim 1. some finite extension of K admits a degree p Galois defect extension.

Indeed, K admits a defect extension L. Since this is defect, $p^n \mid [L:K]$ for some $n \geq 1$. As K is perfect, $K \subset L$ is separable, so we let N be the normal hull of L. Now $K \subset N$ is Galois. If $H < Gal(N \mid K)$ is a p-Sylow subgroup, then $L' = N^H$ is such that:

$$\begin{tikzcd} & L \arrow[r, dash] & N \\ K \arrow[ur, dash] \arrow[r, dash] & L' \arrow[ur, dash] & \end{tikzcd}$$

- $L' \subset N$ is a tower of degree p Galois extensions, $L' = L_0 \subset L_1 \subset \dots \subset L_n = N$,
- $p \nmid [L':K]$.

Since $p \nmid [L':K]$, then $K \subset L'$ is defectless.

Thus $L' \subset N$ is defect, and so for some $i \leq n$,
$$L_i \subset L_{i+1}$$
is Galois defect of degree p. $\square$ CLAIM 1

Now, say $K \subset K' \subset L$ is such that $K' \subset L$ is Galois defect of degree p ($\Rightarrow$ immediate).

Claim 2. If there is a non-trivial definable coarsening of v on K', there is one on K.

Let $\varphi(X, \bar{d})$ define such a val. ring on K'. If $[K' : K] = k \geq 1$, then $\phi : K' \cong K^k$ as K-linear spaces (upon fixing a primitive element for $K \subset K'$). Since this is actually an interpret., $\phi(\varphi(K', \bar{d})) = \tilde{\varphi}(K^k, \bar{d}')$ for some $\bar{d}' \in K$.
Now
$$\tilde{\varphi}(K, 0, \ldots, 0, \bar{d}') \subset K$$
defines the required valuation ring on K. $\square$ CLAIM 2

Claim 3. There is a non-trivial definable henselian coarsening of v on K'.

Let $L = K'(\theta)$, where $\theta^p - \theta = c \in K'$. Now, $K' \subset L$ is an independent defect extension, so there is $H < vK'$ s.t., if $\mathrm{Gal}(L|K') = \langle \sigma \rangle$, then
$$\left\{ v\left(\frac{\sigma(f) - f}{f} \right) : f \in L^\times \right\} = \{ \alpha \in vK' \mid \alpha > H \}.$$

Let $\rho : L \cong (K')^p$ be the interpretation as above.

(Note that this uses c as a parameter). Then, if
$$D = \left\{ \frac{\sigma(f) - f}{f} : f \in L^\times \right\},$$
we have that $\rho(D) = \varphi(K^p, c)$ is $\mathcal{L}_{ring}(c)$-definable.
Denote by $\mathcal{O}_H$ the coarsening of v on L corresponding to H. Then,

- $\rho^{-1}(\mathcal{O}_H) \subset K^p$ is $\mathcal{L}_{ring}(c)$-definable.

$$\mathcal{L} = \mathcal{L}_{ring} \cup \{c\}, \quad \mathcal{L}' = \mathcal{L} \cup \{P(\bar{x})\}$$
$$T = Th_{\mathcal{L}}(K, c, \rho^{-1}(\mathcal{O}_H))$$

We take $(M, c', P_1), (M, c', P_2) \models T$. Then,

(a) $X^p - X - c'$ is irreducible over M, and it gives rise to an extension $M \subset N$, which is interpreted back in M via $\tau : N \cong M^p$,

(b) $\mathcal{O}_1 = \tau^{-1}(P_1)$ & $\mathcal{O}_2 = \tau^{-1}(P_2)$ are henselian valn rings in $H_1(N)$,

(c) for $i = 1, 2$, $x \notin \mathcal{O}_i \iff \exists u \in \mathcal{O}_i^\times$ and $y \in \varphi(M^p, c')$ s.t. $x = u \cdot \frac{1}{\tau^{-1}(y)}$.

As before, $\mathcal{O}_1$ & $\mathcal{O}_2$ are comparable, hence WMA $\mathcal{O}_1 \subset \mathcal{O}_2$. Now we argue exactly as before.
(Beth definability) $\Longrightarrow \rho^{-1}(\mathcal{O}_H) = \tilde{\varphi}(K^p, c)$.

- then, $\mathcal{O}_H \cap K = \tilde{\varphi}(K, 0, \dots 0, c)$. $\square$

CLAIM 3

$\blacksquare$

Remark. We can usually assume $v_K \in H_1(K)$: the case where $v_K \in H_2(K)$ is covered already in full generality by Jahnke & Koenigsmann.

IF TIME PERMITS.

<u>Proposition.</u> K henselian perfect, $\mathrm{char}(K)=p>0$
If

- vK is divisible,
- (K, v_K) is defectless,
- $\exists L \equiv Kv_K$ with (L, v_L) not defectless,

then K admits a $\emptyset$-definable non-trivial hens. val.

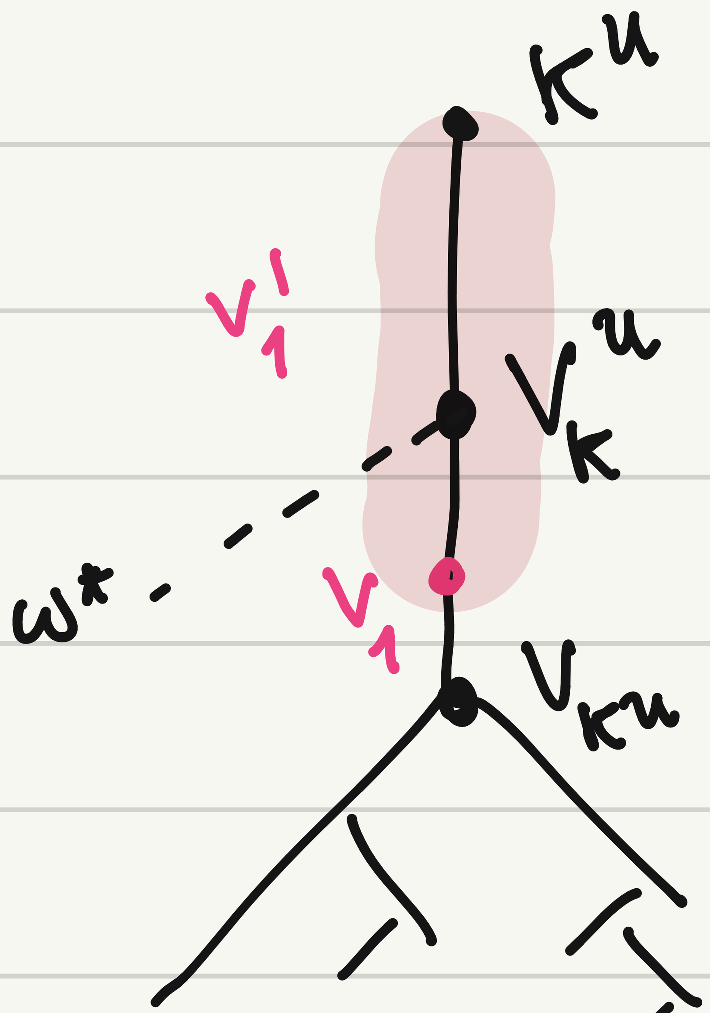
PROOF. WMA $v_K \in H_1(K)$.
Using Keisler-Shelah, there are an index set I and an ultrafilter $\mathcal{U}$ s.t. the ultrapowers $Kv_K^{\mathcal{U}} \cong L^{\mathcal{U}}$.
Let $(K^{\mathcal{U}}, v_K^{\mathcal{U}}) := (K, v_K)^{\mathcal{U}}$, $(L^{\mathcal{U}}, v_L^{\mathcal{U}}) := (L, v_L)^{\mathcal{U}}$.

$$\begin{tikzcd} K^{\mathcal{U}} \arrow[r, "\substack{v_K^{\mathcal{U}} \\ \text{defectless}}"'] \arrow[rr, bend left, "\substack{v_1 \ \text{defect}}"] & Kv_K^{\mathcal{U}} \arrow[r, "\substack{v_L^{\mathcal{U}} \\ \text{defect}}"'] & L^{\mathcal{U}}v_L^{\mathcal{U}} \end{tikzcd}$$

Thus there is a non-trivial $\mathcal{L}_{ring}(K^{\mathcal{U}})$-definable coarsening v_1' of v_1, say $\mathcal{O}_{v_1'} = \varphi(K^{\mathcal{U}}, \bar{\beta})$, $\bar{\beta} \in K^{\mathcal{U}}$.

CLAIM 1. there is an $\mathcal{L}_{ring}(K)$-definable valuation w on K (not nec. henselian) with $\mathcal{O}_w \subset \mathcal{O}_{v_K}$.

This is (Jahnke, 2019). Write $\mathcal{O}_w = \varphi(K, \bar{c})$. $\square_1$



Now, if $\mathcal{O}_{w^*} = \varphi(K^{\mathcal{U}}, \bar{c})$, then w^* is still a proper refinement of $v_K^{\mathcal{U}}$.
Let
$$X = \{\bar{b} \in K^{\mathcal{U}} \mid \varphi(K^{\mathcal{U}}, \bar{b}) \text{ is a non-trivial val. ring}\},$$

which is $\mathcal{L}_{ring}$-definable.
Then, for any $\bar{b} \in X$, the definable val. ring
$$\mathcal{O}_{\bar{b}} := \mathcal{O}_{w^*} \cdot \varphi(K^u, \bar{b})$$
is a coarsening of $\mathcal{O}_{w^*}$ and thus, for any $\bar{b}$ and $\bar{b}'$ in X, $\mathcal{O}_{\bar{b}}$ and $\mathcal{O}_{\bar{b}'}$ are comparable.
In particular, they are all comparable with $\mathcal{O}_{\bar{\beta}}$, which is henselian, and thus with $\mathcal{O}_{v_K^u} = \mathcal{O}_{w^*} \cdot \mathcal{O}_{v_K^u}$.

CLAIM 2. for all $\bar{b} \in X$, $\mathcal{O}_{\bar{b}} \subset \mathcal{O}_{v_K^u}$.

Suppose not, i.e. $\mathcal{O}_{\bar{b}} \supsetneq \mathcal{O}_{v_K^u}$. The corresponding valuation $v_{\bar{b}}$ is a coarsening of v_K^u and thus corresponds to some $H_{\bar{b}} \lneq v_K^u K^u$ (which is divisible).
As (K^u, v_K^u) is tame, we know by Jahnke-Simon that
" if $(K,v) \leq (L,v)$ is saturated and $a, b \in vL$,
$$a \equiv_{vK} b \Rightarrow a \equiv_K b."$$
This implies that $H_{\bar{b}}$ must be $\mathcal{L}_{oag}(v_K^u K^u)$-definable, but $v_K^u K^u$ is divisible. $\square_2$

Now, $\mathcal{O} := \bigcup_{\bar{b} \in X} \mathcal{O}_{\bar{b}} \subset \mathcal{O}_{v_K^u}$ and thus it is non-trivial.
Moreover, it is $\emptyset$-definable & henselian.
$\Rightarrow$ we can define it on K. $\blacksquare$