DEFINING HENSELIAN VALUATIONS:
not all defect is created equal

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§. motivation

Fix a prime $p \in \mathbb{P} = \{2, 3, 5, 7, \dots\}$. Given any $a \in \mathbb{Z} \setminus \{0\}$, we can define

$$v_p(a) := \max \{ n \in \mathbb{N} \mid p^n \text{ divides } a \}.$$

Then, if $a, b \in \mathbb{Z} \setminus \{0\}$ are coprime,

$$v_p\left(\frac{a}{b}\right) := v_p(a) - v_p(b) \in \mathbb{Z}.$$

Now, along with $|0|_p := 0$,

$$|x|_p := p^{-v_p(x)} \in \mathbb{R}_{>0}, \quad x \in \mathbb{Q} \setminus \{0\}$$

defines an absolute value on $\mathbb{Q}$.
Completing $(\mathbb{Q}, |\cdot|_p)$ gives $(\mathbb{Q}_p, |\cdot|_p)$, which is still a field – the p-adic numbers.
Inside of $\mathbb{Q}_p$ there is a "special" subring

$$\mathbb{Z}_p := \{ x \in \mathbb{Q}_p \mid |x|_p \leq 1 \}$$

– the p-adic integers.

Fact. (J. Robinson) $p \neq 2$

$$\mathbb{Z}_p = \{ x \in \mathbb{Q}_p \mid \exists Y (1 + px^2 = Y^2) \}.$$

In other words, $\mathbb{Z}_p$ is $\mathcal{L}_{ring}$-definable.
$\{+, \times, -, 0, 1\}$

⇝ (naive) question: given a field K with absolute value $|\cdot|$, when is
$$\{ x \in K \mid |x| \leq 1 \}$$
an $\mathcal{L}_{ring}$-definable subset?

⚠ PROBLEM: (vaguely)
Absolute values have codomain in $\mathbb{R}$. But when doing model theory, we would like to move to elementary extensions; however, often an el. ext.^{n} in some "reasonable" language $(K, |\cdot|) \preceq (K^*, |\cdot|^*)$ will add infinitesimal elements to $|K^*|^*$, and thus usually $|K^*|^* \nsubseteq \mathbb{R}$.

$\Rightarrow$ we need to allow for non-arch. situations.

§. VALUATIONS

Def.^{n} K a field, $(\Gamma, +, \leq, 0)$ ord. ab. gp
∞ symbol with $\infty > \Gamma$

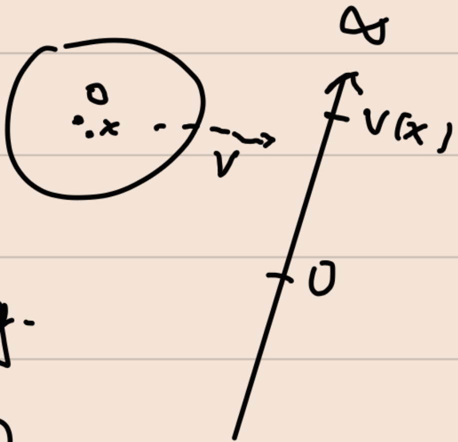
A valuation with value group Γ is a surjective map $v : K \twoheadrightarrow \Gamma \cup \{\infty\}$ such that:

1. $v(x) = \infty \iff x = 0$,
2. $v(xy) = v(x) + v(y)$, $x, y \neq 0$
3. $v(x+y) \geq \min(v(x), v(y))$, $x, y \neq 0$.

Intuition: x "close" to 0

$\iff$

$v(x) \in \Gamma$ big.



Examples. v_p on $\mathbb{Q}$, also on $\mathbb{Q}_p$, both with value gp $\Gamma = \mathbb{Z}$; but we can also have valuations with $\Gamma = \mathbb{Z} \otimes_{lex} \mathbb{Z} \not\leq \mathbb{R}$, etc ...

Note: if (K, v) is a valued field, often the value gp Γ will be denoted by vK (and ∞ will be omitted).

Def.^{n} (K, v) valued field

$$\mathcal{O}_v := \{x \in K \mid v(x) \geq 0\}$$

the valuation ring

$\cup$

$\mathfrak{m}_v := \{ x \in K \mid v(x) > 0 \}$

(unique) max. ideal

$\Rightarrow$ $Kv := \mathcal{O}_v / \mathfrak{m}_v$ - the residue field

<u>Examples.</u> for $(\mathbb{Q}_p, v_p)$, $\mathcal{O}_{v_p} = \mathbb{Z}_p$.

$\leadsto$ (less naïve?) question: given a field K, when is there a $\mathcal{L}_{ring}$-definable valuation ring?

Still a bit too broad / not very interesting.

<u>Def.</u>^{n} a valuation v is henselian if there is a unique valuation $\tilde{v}$ on $\overline{K}$ such that $\tilde{v}|_K = v$. K is henselian if there is atleast one non-trivial henselian valuation on K.

We will use v & $\mathcal{O}_v$ interchangeably.

$\leadsto$ question: given an henselian field K, when is there a $\mathcal{L}_{ring}$-definable non-trivial henselian valuation? $(\ast)$

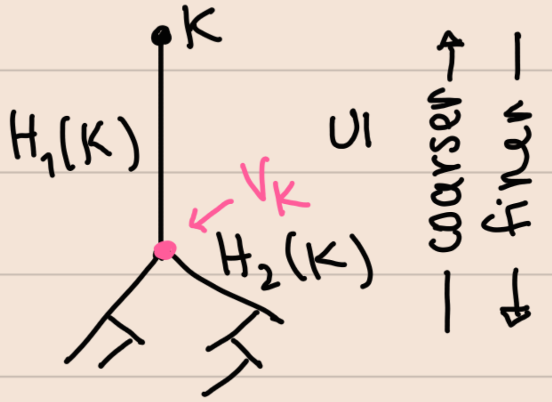
§. answers.

Assume K henselian & not sep. closed.

<u>Def.^{n}</u> $H_1(K) := \{ \mathcal{O}_v \subseteq K : v$ henselian & $Kv \neq Kv^{sep} \}$,

($H_1(K)$ linearly ordered by $\subseteq$)

$H_2(K) := \{ \mathcal{O}_v \subseteq K : v$ henselian & $Kv = Kv^{sep} \}$.



v_K is the canonical henselian val.^{n}

We let v_K be either:
- the finest val.^{n} in H_1, if $H_2 = \emptyset$,
- the coarsest in H_2, o/w.

<u>Philosophy.</u> the answer to $(\star)$ is controlled by the properties of v_K.

<u>Note</u>: this is a general phenomenon in valued fields!

<u>Theorem.</u> (Jahnke–Koenigsmann, 2017; Ketelsen – R. – Szewczyk, 2023+)

K henselian, not sep. closed, perfect.

If $\mathrm{char}(Kv_K) = p > 0 = \mathrm{char}(K)$, then further assume $\mathcal{O}_{v_K}/p\mathcal{O}_{v_K}$ is semi-perfect. Then,

K admits a non-trivial $\mathcal{L}_{\text{ring}}$-definable henselian valuation ring

$$\Longleftrightarrow$$

(i) $Kv_K = Kv_K^{sep}$, OR

(ii) Kv_K is not t-henselian, OR

(iii) $\exists L \succeq Kv_K$ henselian, $v_L L$ not divisible, OR

(iv) $v_K K$ not divisible, OR

(v) (K, v_K) not defectless, OR

(vi) $\exists L \succeq Kv_K$ henselian, (L, v_L) not defectless.

If $\mathrm{char}(Kv_K) = 0$, only these 4 are relevant.

My goal today: show how to use (v).

§. defect.

(K, v) henselian, $K \subseteq L$ finite field extension, then

$$[L:K] \geq (wL:vK)\,[Lw:Kv]. \quad (\dagger)$$
$$\overset{\shortparallel}{\dim_K(L)} \qquad\qquad \overset{\shortparallel}{|wL/vK|}$$

<u>Def.^{n}</u> $(K,v)\subseteq(L,w)$ is defectless if $(\dagger)$ is an equality. (K,v) is defectless if all finite ext^{n}s are.

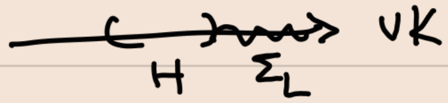
<u>Idea:</u> defect (i.e., not being defectless) is usually "bad". For us, however, it is a source of information.

<u>Def.^{n}</u> $(K,v)\subseteq(L,w)$ defect, Galois of degree $p=\mathrm{char}(Kv)$. Then $\mathrm{Gal}(L|K)=\langle\sigma\rangle$.
Let $\Sigma_L := \left\{ v\left(\frac{\sigma(f)-f}{f}\right) : f\in L^\times \right\} \subseteq wL.$

Indeed, we may assume $vK=wL$ and thus have $\Sigma_L\subseteq vK$.

<u>Def.^{n}</u> $(K,v)\subseteq(L,w)$ as above has indep. def. if there is $H\leq vK$ convex s.t.

- $\Sigma_L = \{\alpha\in vK \mid \alpha > H\}$,
- vK/H has no smallest pos. element.



Under our hypotheses, we will often be in a situation where the extension is an ind. defect one. Now, Σ_L is "essentially" $\mathcal{L}_{ring}$-definable, thus so is H & the valuation corresponding to

$$\begin{tikzcd} K \arrow[r, "v"'] \arrow[rr, bend right, "\tilde{v}"'] & vK \arrow[r, two heads] & vK/H. \end{tikzcd}$$

<u>Q^n:</u> examples?