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Étale methods in model theory

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A nonno Dante e nonno Clemente.

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Prologue

Some humans would do anything to see if it was possible to do it. If you put a large switch in some cave somewhere, with a sign on it saying

"*End-of-the-World Switch.*
PLEASE DO NOT TOUCH"

the paint wouldn't even have time to dry.

Terry Pratchett, *Thief of Time*

IT would be hard to introduce this story, to open the curtains on this thesis, without some context on what the world was like throughout its birth. I would like to paint some details that might, or might not, justify some choices. The topic was chosen, among a list of many, somewhere in Central Italy, close to the sea, during a time where the pandemic seemed somehow more distant, less frightful; there were plans, at that point, of moving across the continent for some months. These plans eventually – but ever so slowly – came to a natural death as the months progressed and the pandemic worsened. Of those five months, only ten days survived. I will have to find a good way to add "Erasmus program" to my CV, when the most I have travelled while writing this thesis was from my bedroom to my kitchen, and back. Despite all of the difficulties of writing a thesis from abroad, it was nevertheless a vivid experience – beautiful, frustrating and educational. There are many people who have to be thanked for making the six-ish months of this thesis as filled with stories to be told as they were, but that will have to wait until the Epilogue; for now, I have spoken enough and it is high time I leave this stage to the only worthy actor of this ensemble: the story itself.

THERE are several possible ways to start this story, all of which are necessarily apocryphal. What are "étale methods", and why do we care? Or, more precisely, why do *I* care? Here is one possible explanation: in 1989, Van den Dries showed in [Dri] that definable sets, in henselian valued fields of characteristic zero, admit a particularly nice description, that somehow resembles the one available for definable sets in algebraically closed fields. In particular,

THEOREM. — *If k is a henselian valued field of characteristic zero, every definable set can be written as a finite union of valuation open subsets of Zariski closed sets.*

A result like this one begs the question – can something similar be said about other classes of fields? Can one find a unifying framework encompassing all fields (possibly with some further hypotheses) whose definable sets can be decomposed this way? The first step is, necessarily, finding an adequate notion of "open" – such a notion should, then, survive a reality check: it should coincide with available, *classical* notions of openness, like Zariski open or valuation open, in all natural cases.

IN come étale methods. Or more specifically, the étale-open topology, which is a candidate – introduced in [Joh+20] – for the notion of "openness" that was needed. One might think of it as a way of assigning, uniformly, a topology to all k-varieties over a certain field k; and indeed, it is the data of a covariant functor $\mathcal{E}_k : (\mathbf{Var}_k) \to (\mathbf{Top})$ that *carries over* geometrical (and algebraic) information from the k-varieties to certain topological spaces obtained from their k-points.

THE étale-open topology is two-faced – on the one hand, $\mathcal{E}_k$ acts as a *dictionary* between algebraic properties of k and topological properties of $\mathcal{E}_k$. As an example, the field k is *not* separably closed if and only if $\mathcal{E}_k$ is Hausdorff on every quasi-projective k-variety. On the other, it generalizes several well-known topologies: if k is separably closed, for example, $\mathcal{E}_k$ is just the Zariski topology; if k is real closed, it is the order topology; if it is henselian, it is the valuation topology.

BOTH of these faces come into play when the étale-open topology is applied to obtain results on the algebraic properties of certain model-theoretically interesting fields. In chapter 3 and 4, two of these kind of applications are explored – under the assumption of *largeness*, without which the étale-open topology is just the discrete topology: a specific instance of the Stable Fields Conjecture is proved, namely that stable large fields are separably closed; and on a similar vein, in the following chapter, simple large fields are shown to be bounded.

FINALLY, chapter 5 goes back to the beginning of our story. Now that we have a notion of "openness" at hand, we can isolate the class of fields where definable sets admit a nice topological decomposition, namely *éz* (pronounced like "easy") fields. In particular, two examples of éz fields are explored in detail: henselian valued fields of characteristic zero, and algebraically maximal Kaplansky valued fields.

⚠ There are several places where there might be useful remarks, or historical notes, that would break down the flow of the exposition. In that case, I will put them in small, grey boxes, like

> ✱ This one.

1.1 What to expect

THERE are several choices that have been made while writing this thesis. The first, and biggest, one was to put the emphasis on the exposition and the ideas, rather than *all* the technical details. For this reason, there will be several facts without proofs – the idea being that those proofs wouldn't contribute to the final understanding of the content. In particular, several proofs from the sections on stability and simplicity have only been sketched, either because they weren't particularly interesting *or* because the machinery involved would take too long to be introduced, distracting the reader from the final goal. At any rate, references are always given, so that those who feel like they want more details can find them (and possibly explain them to me). The moral of this story is twofold: one, hopefully nobody will try to learn stability or simplicity from my thesis; two, trusting your chaperone on this journey – that is, me – is a necessary act of faith, possibly one that won't leave you (too) disappointed.

1.2 Notation

$\mathbb{N}$	the natural numbers, *including zero* (alternatively, ω)
KF	the compositum of the fields K and F (inside a common extension)
$\cong_k$	isomorphism of fields over a common subfield k
K^h	the henselianization of the (valued) field K
K^{alg}	the algebraic closure of the field K
K^{sep}	the separable closure of the field K
$K((t^\Gamma))$	the field of Hahn series with coefficients in K and exponents in the group Γ; if Γ is omitted, assume $\Gamma = \mathbb{Z}$
$K^I/\mathcal{U}$	the ultrapower of the field K over the ultrafilter $\mathcal{U}$ on the set I
L_{ring}	the language of rings $\{0, 1, +, \cdot\}$

Étale methods

La luna geme sui fondali del mare,
o Dio quanta morta paura
di queste siepi terrene,
o quanti sguardi attoniti
che salgono dal buio a ghermirti nell'anima ferita.

Alda Merini, *Canto alla luna*

EVERY story begins with a promise. I must, therefore, proceed to promise you something, a reward that will make working through these pages worth the effort. The starting point is that mathematics, as it is taught and done, is effectively an oscillating Babel tower, a place where dozens of different languages are spoken, all at the same time, all at the maximum volume possible. Exploring such a labyrinth of voices, where new sounds and words are created all the time, is a struggle that is ever so slightly reduced by the introduction of dictionaries and translations. The first ones are taught in high school: lines in the plane correspond to linear equations. Parabolas to quadratic equations. Jumping ahead, algebraic geometry works as a translation paradigm between algebra and geometry. Many other similar *dictionaries* have been, and are being, discovered throughout mathematics.

THIS is my promise to anyone reading this introduction: in these pages, you will find a new dictionary, a bridge between vastly different worlds. The étale-open topology will allow communication between people from the topological mountains and inhabitants of the algebraic fjords.

I can only hope I can maintain this promise.

2.1 Large fields

BEFORE moving on to the central topic of this chapter, we need a short detour into the theory of *large* fields. As introduced by Pop in 1996, they are a class of fields that exhibit interesting behaviours from many points of view – model theoretic, Galois theoretic, diophantine. The definition somehow aims at capturing this variety of phenomena, isolating fields that are *rich* enough for fruitful interactions between algebra and geometry to happen. By a k-curve we will mean a reduced k-scheme of finite type and dimension 1.

Definition 2.1.1. — A field k is said to be **large** if every absolutely irreducible k-curve C with one smooth k-rational point has infinitely many k-rational points.

MANY fields that can be found in nature end up being large. For example, algebraically closed fields, real closed fields, PAC (and hence pseudofinite) fields, henselian valued fields are all large. On the contrary, as a consequence of the Mordell conjectures, number fields – including $\mathbb{Q}$ – are not large. This often leads to calling them "anti-Mordellic."

✱ It seems like "anti-Mordellic" is also a clever joke in French, where *bordélique* means "messy" in slang.

THERE are several fields which are not known to be large, nor there is any hint towards their largeness (or non-largeness). Among them, the most notable is arguably $\mathbb{Q}^{ab}$, which is the largest abelian extension of $\mathbb{Q}$ – by Kronecker-Weber, this is just $\mathbb{Q}$ adjoint with all primitive roots of unity. Despite this concrete description, a proof of its largeness – or non-largeness – is still elusive, and in fact it would imply several interesting conjectures in Galois theory, see for example [BF11].

THE definition of large was given in terms of curves, but there is really no need for that; through a smooth k-rational point of any k-variety one can find many curves, all of which will have infinitely many k-points, and thus one can actually prove that the definition of large is equivalent to a higher dimensional version of it.

✱ This is, at least philosophically, a consequence of Bertini's theorem.

PROPOSITION 2.1.2. — *The following are equivalent:*

1. *k is large,*
2. *every absolutely irreducible k-variety V of positive dimension with a smooth k-rational point has infinitely many k-rational points,*
3. *if V is an absolutely irreducible k-variety of positive dimension with a smooth k-point, then $V(k)$ is Zariski dense in V.*

BEFORE moving on, a small remark on the elementarity of largeness. Elementarity is more or less easily achievable once we can restrict the set of curves we have to check. In particular, we can restrict to checking planar curves, thanks to this lemma (see [Pop13], Fact/Notations I).

LEMMA 2.1.3. — *Suppose X is an irreducible k-variety of dimension d and p is a smooth point of X. Then there is an hypersurface $X_0 = V(f) \subseteq \mathbb{A}^{d+1}$ and a birational equivalence $\varphi : X \dashrightarrow X_0$ defined at p and such that $\varphi(p) = (0,\ldots 0)$. The polynomial f can be taken of the form $f = X_{d+1} + \hat{f}$, where $\hat{f} \in k[X_1,\ldots X_{d+1}]$ has vanishing terms of degree less than two. Moreover, $X(k) \subseteq X$ is Zariski dense if and only if $X_0(k) \subseteq X$ is Zariski dense.*

AS a consequence, we can prove that

COROLLARY 2.1.4. — *The class of large fields is elementary in the language of rings.*

DENOTE by X_{sm} the set of smooth points of X.

Proof. We shall see that k is large if and only if, for every irreducible polynomial $f = X_2 + \sum_{i+j>1} a_{ij} X_1^i X_2^j$ and every finite $S \subseteq k$, there is $(a,b) \in k^2$ such that $f(a,b) = 0$ and $a \notin S$. The latter is an elementary property. Assume k is large. Then $C = V(f) \subset \mathbb{A}^2$ has $(0,0)$ as a smooth point. Either $f = X_2$, or f is not constant in X_1; in both cases, the condition is satisfied. Now assume k satisfies the latter condition. Let C be a k-curve with a smooth point $p \in C(k)$. We can, by the lemma just mentioned, choose C_0 birationally equivalent to C via φ, $C_0 = V(f)$ where f has the form

$$f = X_2 + \sum_{i+j>1} a_{ij} X_1^i X_2^j.$$

Let $U \subseteq C$ and $U_0 \subseteq C_0$ be the Zariski open subsets isomorphic via φ: since $p \in U$, we can assume $U \subseteq C_{\mathrm{sm}}$ and thus $U_0 \subseteq (C_0)_{\mathrm{sm}}$. Further, $C \setminus U$ and $C_0 \setminus U_0$ are finite, hence if $C_0(k)$ is infinite, $C(k)$ is also infinite. ∎

2.1.1 *Largeness and existential closure*

ONE further source of interest over large fields is the fact that they can be isolated using purely model theoretical concepts. For starters, we need to recall some notation.

Definition 2.1.5. — Let M and N be two structures in the same language L, with $M \subseteq N$. We say that M is **existentially closed in N**, and write $M \prec_\exists N$, if for every quantifier-free L-formula $\varphi(x)$, if $N \models \exists x \varphi(x)$ then $M \models \exists x \varphi(x)$.

WE now focus on fields. Existential closure satisfies important properties. In particular, it is a transitive and hereditary relation.

THEOREM 2.1.6. — *Suppose $k \subseteq K \subseteq L$ is a tower of field extensions. On the one hand, if $k \prec_\exists K \prec_\exists L$, then $k \prec_\exists L$. On the other hand, if $k \prec_\exists L$, then $k \prec_\exists K$.*

MOREOVER, existential closure can be characterized through ultrapowers.

THEOREM 2.1.7. — *Suppose $k^I/\mathcal{U}$ is an $|K|$-saturated ultrapower of k. Then $k \prec_\exists K$ if and only if there exists a field embedding $K \subseteq k^I/\mathcal{U}$ over k.*

IN the class of fields, we are now ready to prove that large fields are exactly those that are existentially closed in their fields of Laurent series. We will follow chapter 5 of [Jar11] for this proof.

THEOREM 2.1.8. — *k is large if and only if $k \prec_\exists k((t))$.*

WE prove this through several lemmas. Here is the strategy.

1. We prove that if k is large, then function fields of one variable over k with one k-rational place have infinitely many such places.
2. Using this, we prove that k is existentially closed in the henselianization of $k(t)$.
3. Since $k(t)^h$ is existentially closed in its completion, which we can assume is $k((t))$, we have showed one implication.
4. Finally, if we assume that k is existentially closed in $k((t))$, we prove directly that it is large (using the henselianity of the t-adic valuation).

BEFORE moving forward with the proof of 2.1.8, we recall some definitions and facts from the theory of function fields of one variable. We draw heavily from Chapter 3 of [FJ08].

Definition 2.1.9. — If k is a field, a **function field of one variable** over k is a finitely generated regular extension F with transcendence degree 1. A **model** of the field extension $k \subseteq F$ is a k-curve C such that $k(C) \cong_k F$.

IF F is a function field of one variable, then there is $t \in F$ which is trascendental over k and such that $k(t) \subseteq F$ is finite and separable.

LEMMA 2.1.10. — *Suppose k is large, then every function field of one variable over k with a k-rational place*[1] *has infinitely many k-rational places.*

Proof. By Lemma 5.1.4 in [Jar11], we can find an affine model C for $k \subseteq F$ that has a smooth k-rational point p. By largeness, C has infinitely many smooth k-rational points. Each one of them gives, again by Lemma 5.1.4, a k-rational place on F (which is precisely the one corresponding to the valuation ring $\mathcal{O}_{C,q}$ for every k-rational point q). ■

THERE is a natural valuation on $k(t)$, namely the discrete one that is trivial on k and assigns value 1 to t. We call this the **t-adic** valuation, and will always assume it is the valuation we consider on $k(t)$. Similarly for $k((t))$, where the t-adic valuation is the canonical henselian valuation. We shall denote by $k(t)^h$ the henselianization of $k(t)$, which we see as a subfield of $k((t))$.

[1] A place is called k-rational if the residue field is exactly k.

LEMMA 2.1.11. — *Suppose k is large, then $k \prec_{\exists} k(t)^h$.*

Proof. Let $x = (x_1, \dots x_n)$ be a point with coordinates in $k(t)^h$. Let $F = k(x)$: if x were algebraic over k, then because k is algebraically closed in $k((t)) \supseteq k(t)^h$ we must have $x \in k^n$, in which case we are done, since x satisfies the same equations as itself. Otherwise, the transcendence degree of $k \subseteq F$ is 1. Let φ_0 be the restriction to F of the place corresponding to the t-adic valuation on $k((t))$: it is k-rational and so, by hypothesis, $k \subseteq F$ admits infinitely many k-rational places. Apart from a finite set of them, they will all be finite at x, so the specializations will stay in k^n, hence $k \prec_{\exists} k(t)^h$. ■

WE can finally prove 2.1.8.

Proof. In one direction, since $k \prec_{\exists} k(t)^h$ and $k(t)^h \prec_{\exists} k((t))$, then $k \prec_{\exists} k((t))$.

✱ The fact that $k(t)^h \prec_{\exists} k((t))$ is an instance of the fact that the henselianization of a function field in one variable is existentially closed in its completion, which we can assume is $k((t))$ in this case. See Lemma 5.2.7 in [Jar11].

In the other direction, suppose $f \in k[x, y]$ is an absolutely irreducible polynomial and $(a, b) \in k^2$ is a point such that $f(a, b) = 0$ and $\frac{\partial f}{\partial y}(a, b) \neq 0$. We prove that the k-variety $A = V(f)$ has infinitely many k-points, thus proving that k is large (see the proof of 2.1.4). Let v be the t-adic valuation on $k((t))$ and suppose $(a_1, b_1), \dots (a_n, b_n) \in A$. Note that, for a choice of a' in $k((t))$ such that $a' \neq a_1, \dots a_n$ such that $v(a' - a)$ is big enough, $v\left(\frac{\partial f}{\partial y}(a', b)\right) = v\left(\frac{\partial f}{\partial y}(a, b)\right)$ is finite and

$$v(f(a', b)) = v(f(a', b) - f(a, b)) > 2v\left(\frac{\partial f}{\partial y}(a', b)\right)$$

and so, by henselianity, we can find $b' \in k((t))$ such that $f(a', b') = 0$. Now since k is existentially closed in $k((t))$, we can find points a_{n+1} and b_{n+1} in k that kill f, and $a_{n+1} \neq a_1, \dots a_n$. This shows that A contains a new point, (a_{n+1}, b_{n+1}). Since this can be iterated, A is infinite. ■

2.1.2 *Fraction fields of henselian domains*

BEFORE moving on to étale methods, we stop for a moment to examine thoroughly one of the examples of large fields mentioned at the beginning.

THE main source for this section will be [Pop10]. Instead of looking precisely at henselian valued fields, we can check that something slightly more general is large.

Definition 2.1.12. — A ring R will be called **henselian with respect to $\mathfrak{a}$**, where $\mathfrak{a}$ is some ideal, if the following holds: if $a + \mathfrak{a} \in R/\mathfrak{a}$ is a root of some polynomial $\overline{f}(X) \in R/\mathfrak{a}[X]$ such that $\overline{f}'(a + \mathfrak{a}) \in (R/\mathfrak{a})^\times$, then there exists $b \in R$ such that $f(b) = 0$ and $b + \mathfrak{a} = a + \mathfrak{a}$.

IN particular, if (k, v) is a henselian valued field, then $\mathcal{O}_v$ with $\mathfrak{m}_v$ is a henselian ring.

THEOREM 2.1.13. — *If R is a domain which is henselian with respect to some ideal $\mathfrak{a} \neq (0)$, then $k = \mathrm{Quot}(R)$ is a large field.*

Proof. Suppose C is an absolutely irreducible k-curve and $p \in C(k)$ is a smooth k-rational point. We can assume, up to birational equivalence, that $C = V(f) \hookrightarrow \mathbb{A}^2_k$, p is identified with the origin and the polynomial has the form $f(X,Y) = Y + \hat{f}(X,Y)$, where $\hat{f}$ has vanishing terms in degree less than two. After clearing denominators, we can assume $\hat{f}(X,Y) \in R[X,Y]$, so that we have a plane curve $\tilde{C} := \mathrm{Spec}\, R[X,Y]/(f) \subseteq \mathbb{A}^2_R$.

For every $a \in \mathfrak{a}$, set $f_a(Y) = f(a,Y)$. Then $f_a(0) = f(a,0) \in \mathfrak{a}$ and $f'_a(0) \in 1+\mathfrak{a}$, so $0+\mathfrak{a}$ is a simple root of $\overline{f}_a \in (R/\mathfrak{a})[Y]$. By henselianity, there exists an element $b = b(a) \in \mathfrak{a}$ such that $f_a(b) = f(a,b) = 0$, i.e. $(a,b) \in \tilde{C}(R) \subseteq C(k)$. This defines an injection $\mathfrak{a} \hookrightarrow C(k)$ given by $a \mapsto (a, b(a))$, hence $|C(k)| \geq |\mathfrak{a}| \geq \omega$. ■

2.2 Étale maps

ÉTALE maps will constitute the bread and butter of the techniques we will use later on, as developed in [Joh+20]. There are several possible references for étale maps and techniques, none of which are exactly what we need. Mostly, we will refer to [Jon], which is an excellent reference but perhaps not the best place to start learning étale methods from; Milne's notes ([Mil13]) are written in his well-known clear style, but cover mostly the classical case, while his 80s textbook ([Mil80]) covers the general case; at any rate, one can find almost anything in [Gro64b] (or the amazing community translation, [HK20]).

✱ According to Milne, the word "étale" is intended by Grothendieck in the way Hugo uses it in "*La mer était étale*, mais le reflux commencait a se sentir", i.e. to mean the sea at high or low tide. His intuition was that étale maps were like the light of a full moon on a sea at high tide: locally parallel, but not globally so.

THERE are several equivalent definitions of étale morphisms, none of them particularly illuminating. I chose this one, because it is a good working definition, but it might not be the most popular.

Definition 2.2.1. — A morphism of schemes $\varphi : X \to Y$ is **étale at p** if it is smooth of relative dimension 0 at p. A morphism of schemes $X \to Y$ is **étale** if it is étale at every point.

THIS means that, for every point $p \in X$ and affine open neighbourhood $U \subseteq X$ of p there is an affine open neighbourhood $V = \mathrm{Spec}\, R$ of $\varphi(p)$ in Y such that "$\varphi|_U$[2] is like (the map induced by) the map from a ring to a quotient of its ring of polynomials".

[2] Or, more precisely, $\varphi|_{U \cap \varphi^{-1}(V)}$.

IN other words, there is $n \geq 0$ and polynomials $f_1, \dots f_n \in R[x_1, \dots x_n]$ such that U is isomorphic to an open subscheme W of

$$\operatorname{Spec} R[x_1, \dots x_n]/(f_1, \dots f_n)$$

and the following diagram commutes,

$$\begin{array}{ccccc} U & \xrightarrow[\cong]{} & W & \hookrightarrow & \operatorname{Spec} R[x_1, \dots x_n]/(f_1, \dots f_n) \\ \downarrow{\scriptstyle \varphi|_U} & & \downarrow{\scriptstyle \rho|_W} & \swarrow{\scriptstyle \rho} & \\ V & \xrightarrow{\cong} & \operatorname{Spec} R & & \end{array}$$

where ρ is the map induced by the ring map

$$R \to R[x_1, \dots x_n]/(f_1, \dots f_n)$$

and the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j \leq n}$ has rank n.

THE definition of étale is somewhat obscure. The intuition to keep in mind is as follows: in differential geometry, one has the idea of "local diffeomorphism", meaning a map $\varphi : M \to N$ between smooth manifolds such that, when one takes a neighbourhood $U \subset M$ of $p \in M$ and a neighbourhood $V \subset N$ of $\varphi(p) \in N$ such that $\varphi[U] \subseteq V$, $\varphi|_U : U \to V$ is a diffeomorphism onto its image. This, in particular, implies that $\varphi_{*,p} : T_p M \xrightarrow{\sim} T_{\varphi(p)} N$. The viceversa is precisely the implicit function theorem – or rather, the local inverse theorem. Suppose now that you are given "classical" affine varieties $V, W \subseteq \mathbb{A}^n$, that is good old sets of zeroes of polynomials over an algebraically closed field k. Given a regular map $\varphi : V \to W$, one can consider its differential $\varphi_{*,p} : T_p V \to T_{\varphi(p)} W$ at a point $p \in V$ and wonder if, by imposing that $\varphi_{*,p}$ be an isomorphism, it is possible to recover an implicit function theorem. That does not work – the Zariski topology is too coarse for our dreams. Nevertheless, étale maps provide a good counterpart to covering maps, to the extent that one can build a cohomology theory out of them.

WE now explore some of the main properties we will need.

LEMMA 2.2.2. — *Open immersions are étale.*

Proof. Suppose $\iota : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is an open immersion. We can, modulo moving to open subschemes, assume $X \cong \operatorname{Spec} R \cong \iota(X)$. Then we can take

$$W = \operatorname{Spec} R[x_1, \dots x_n]/(x_1, \dots x_n)$$

so that the induced map ρ restricts precisely to ι. ■

LEMMA 2.2.3. — *The composition and base change of étale morphisms is still étale.*

Proof. Showing that the composition of étale morphisms is étale reduces to showing the following algebraic fact: suppose you have a sequence of rings $A \to B \to C$ such that

$$B = \frac{A[x_1, \dots x_n]}{(f_1, \dots f_n)}$$

with $\det\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j \le n}$ non-zero and

$$C = \frac{B[y_1, \dots y_m]}{(g_1, \dots g_m)}$$

where $\det\left(\frac{\partial g_i}{\partial y_j}\right)_{i,j \le m}$ is non-zero. Then we can rewrite C as a quotient of a polynomial ring in A for an ideal of polynomials whose Jacobian has the right rank. To do so, consider that to each $g_i \in B[y_1, \dots y_m]$ is associated a $g'_i \in A[x_1, \dots x_n, y_1, \dots y_m]$, which is given by considering representatives for each of the coefficients of g_i. Then

$$C \cong \frac{A[x_1, \dots x_n, y_1, \dots y_m]}{(f_1, \dots f_n, g'_1, \dots g'_m)}$$

and the combined Jacobian of the f_is and g_js has precisely rank $n + m$.

Consider now an étale map $\varphi : X \to Y$ of Z-schemes. If $Z' \to Z$ is another Z-scheme, then we get a new map

$$\varphi_{Z'} = f \times_Z \mathrm{id}_{Z'} : X \times_Z Z' \to Y \times_Z Z'.$$

By definition, there is an affine open $U \subseteq X$ that is isomorphic to an open subscheme $W \subseteq \operatorname{Spec} R[x_1, \dots x_n]/(f_1, \dots f_n)$ and an affine open $V = \operatorname{Spec} R \subseteq Y$ such that $\varphi|_U$ is isomorphic to the (restriction to W of the) induced map between $\operatorname{Spec} R[x_1, \dots x_n]/(f_1, \dots f_n)$ and $\operatorname{Spec} R$. Modulo moving to affine open subschemes, we can assume $Z' = \operatorname{Spec} S'$ and $Z = \operatorname{Spec} S$. Take $U' = U \times_S Z'$ and $V = V \times_S Z'$, so that

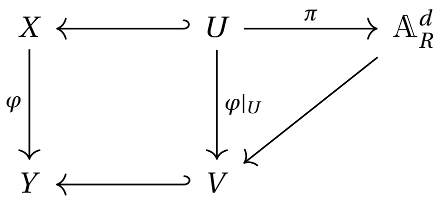
$$U' \cong W \times_S Z' \subseteq \operatorname{Spec} \frac{R[x_1, \dots x_n]}{(f_1, \dots f_n)} \otimes_S S' \cong \operatorname{Spec}(R \otimes_S S')[x_1, \dots x_n]/(\overline{f_1}, \dots \overline{f_n})$$

where $\overline{f_i}$ is obtained from f_i by tensoring each coefficient with 1. Moreover, $V' \cong \operatorname{Spec} R \otimes_S S'$. The Jacobian associated to $\overline{f_1}, \dots \overline{f_n}$ is obtained from the Jacobian associated to $f_1, \dots f_n$ by tensoring each element with 1, so it still has rank n and the map $\varphi_{Z'}$ is étale. ■

✱ This proof is much more powerful, because it allows us to show that if you compose two smooth maps, of relative dimension r and s, you find a smooth map of relative dimension $r+s$. In this case, we are working with $0+0=0$.

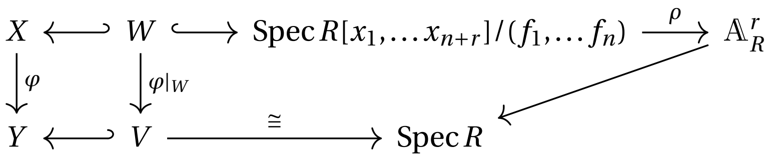
MOREOVER, étale maps provide a powerful intuition for the local properties of smooth morphisms. In particular,

LEMMA 2.2.4. — *Suppose $\varphi : X \to Y$ is a morphism of k-varieties and p is a point in X. If φ is smooth at p, then for every affine open neighbourhood of $\varphi(p)$, $V = \operatorname{Spec} R \subseteq Y$, there is an affine open neighbourhood of p, $U \subseteq X$, and an integer $d \geq 0$ such that*



commutes, where π is étale.

Proof. Suppose φ is smooth of relative dimension r. Then we have, by definition, the following diagram



where the map ρ is induced by quotient projection. Notice that

1. the map $U \to W$ is an isomorphism, hence étale,

2. the map $W \to \operatorname{Spec} R[x_1, \dots x_{n+r}]/(f_1, \dots f_r)$ is an open immersion, hence étale,

so we only need to show that ρ is étale. Notice that ρ is induced by the ring map

$$R[x_1, \dots x_r] \to \frac{R[x_1, \dots x_r][x_{r+1}, \dots x_{n+r}]}{(f_1, \dots f_n)}$$

and the Jacobian $(\partial f_i / \partial x_{r+j})_{i,j \leq r}$ has rank r, so ρ is étale. ∎

✱ Étale morphisms satisfy what Ravi Vakil, in [Vak], defines as the properties of "reasonable" classes of morphisms. They are satisfied by étale morphisms, and *a fortiori* open immersions, and many other classes, and indeed a lot of the work done here could be partially done with any class of morphisms satisfying these formal properties.

IF, in particular, X is a curve, i.e. it is irreducible and it has dimension 1, $Y = \operatorname{Spec} k$ and φ is the structure morphism $X \to \operatorname{Spec} k$, we get an étale morphism $\pi : U \to \mathbb{A}^1_k$. We call this a **local coordinate** at p.

IN the very specific case of $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots x_n]$, we get a very useful criterion. Suppose you have a map of k-varieties, $\varphi : \mathbb{A}^n_k \to \mathbb{A}^n_k$, induced by some $\psi : k[x_1, \dots x_n] \to k[y_1, \dots y_n]$. Let $f_i(y_1, \dots y_n) := \psi(x_i)$ for $i = 1, \dots n$. Then we can rewrite the map $\mathbb{A}^n_k \to \mathbb{A}^n_k$ as a map

$$\operatorname{Spec} k[\overline{x}, \overline{y}]/(x_1 - f_1(\overline{y}), \dots x_n - f_n(\overline{y})) \to \operatorname{Spec} k[x_1, \dots x_n]. \qquad (2.1)$$

THIS allows to turn the intuition that an invertible Jacobian makes a map a local isomorphism – i.e., étale – into a proper criterion. Notice, however, that étale doesn't *really* mean local isomorphism, not even in the modern case. The Zariski topology remains too coarse, and one has to turn to the so-called "étale topology" (which is *not* the étale-open topology, or not even a topology in a classical sense, for that matter).

PROPOSITION 2.2.5. — *Let $\varphi : \mathbb{A}^n_k \to \mathbb{A}^n_k$ be a map of k-varieties. Suppose that, in the notation of 2.1, $\left(\frac{\partial f_i}{\partial y_j}\right)_{i,j \le n}$ is invertible at p. Then φ is étale at p.*

Proof. The map φ can be written locally precisely as

$$\operatorname{Spec} k[\overline{x}][y_1, \dots y_n]/(x_1 - f_1(\overline{y}), \dots y_n - f_n(\overline{y})) \to \operatorname{Spec} k[x_1, \dots x_n].$$

Let $h_i(\overline{x}, \overline{y}) = x_i - f_i(\overline{y})$. Then the Jacobian $\left(\frac{\partial h_i}{\partial y_j}\right)_{i,j \le n} = -\left(\frac{\partial f_i}{\partial y_j}\right)_{i,j \le n}$ has rank n in a neighbourhood of p (since the determinant is a polynomial with coefficients in k, the locus on which it is not zero is open). ■

2.3 The étale-open topology

AS mentioned in the introduction, we will now construct a dictionary between topological and algebraic information. We will consider a field k, and this dictionary will be provided by the data of a certain topology on each k-variety. If we think of étale maps as coverings of varieties, we are essentially building each variety with bricks that look like copies of other varieties (of the same dimension, at least on an irreducible component).

FOR a fixed field k and a k-variety V (not necessarily reduced), an **étale image** in $X(k)$ is a set of the form $\varphi(V(k))$ for some étale morphism of k-varieties $\varphi : X \to V$.

✱ This "not necessarily reduced" is precisely what forces us to use schemes instead of algebraic sets. More specifically, we shall look at fibered products, that can – and usually do – end up being non-reduced.

LEMMA 2.3.1. — *The collection of étale images in $V(k)$ contains every Zariski open subset of $V(k)$ and it is closed under finite unions and intersections.*

Proof. Suppose $U \subseteq V(k)$ is Zariski open: then, there is an open subvariety $W \subseteq V$ such that $W(k) = U$. Since the immersion $\iota : W \to V$ is étale, $U = \iota(W(k))$ is an étale image in $V(k)$. Suppose now $U_1 = f_1(W_1(k))$ and $U_2 = f_2(W_2(k))$ are two different étale images in $V(k)$. Then the canonical map $h : W_1 \times_V W_2 \to V$, respectively $g : W_1 \sqcup W_2 \to V$, is étale and has image $U_1 \cap U_2$, respectively $U_1 \cup U_2$. ■

IN particular,

COROLLARY 2.3.2. — *The collection of étale images is the basis for a topology on $V(k)$.*

CALL this topology the **étale-open topology** on $V(k)$ and denote it by $\mathcal{E}_k(V)$. The following results will show that the assignment

$$V \mapsto (V(k), \mathcal{E}_k(V)),$$
$$(\varphi : V \to W) \mapsto (\varphi : V(k) \to W(k))$$

determines a functor $(\mathbf{Var}_k) \to (\mathbf{Top})$ that carries over some of the geometrical information of V. It will be, in particular, a system of topologies – they will be introduced in 2.4, but for now know that "carries over some geometrical information" means that we want our functor to

1. transform open immersions in (topological) open embeddings,

2. transform closed immersions in (topological) closed embeddings.

BUT first and foremost, we establish that it is actually a functor.

THEOREM 2.3.3. — *For every $\varphi : V \to W$, the induced map*

$$\varphi : (V(k), \mathcal{E}_k(V)) \to (W(k), \mathcal{E}_k(W))$$

is continuous.

Proof. Something slightly stronger is true – suppose $U \subseteq W(k)$ is an étale image. Then $f^{-1}(U) \subseteq V(k)$ is an étale image. In fact, let $U = h(X(k))$ and consider the pullback square

$$\begin{array}{ccc} V \times_W X & \longrightarrow & X \\ {\scriptstyle \psi}\downarrow & \lrcorner & \downarrow{\scriptstyle h} \\ V & \xrightarrow[f]{} & W \end{array}$$

where ψ is the base change of h and is thus étale. In particular, $f^{-1}(U) = \psi((V \times_W X)(k))$ is an étale image. ■

NEXT, we prove something slightly more general than what we need – but since open immersions are étale, this gives us item 1 of our list.

THEOREM 2.3.4. — *If $\varphi : V \to W$ is étale, then*

$$\varphi : (V(k), \mathcal{E}_k(V)) \to (W(k), \mathcal{E}_k(W))$$

is open.

Proof. If $U = h(X(k)) \subseteq V(k)$, then $\varphi(U) = (f \circ h)(X(k))$ is an étale image, hence open. ■

SHOWING that closed immersions are sent to closed embeddings requires some more work, and in particular the following lemma, which was taken from [Gro64b]. It deploys the machinery of "standard smooth" maps, as introduced in appendix A.2.1.

LEMMA 2.3.5. — *If W is a k-variety, V is a closed subvariety of W, X is a k-variety with an étale morphism $g : X \to V$ and p is a point of X, then there is an open subvariety O of X that contains p and such that O is isomorphic, as a V-scheme, to the fiber product $Y \times_W V$ where Y is a k-variety and the fiber product is built using some étale morphism $h : Y \to W$.*

Proof. It is enough to produce a smooth $h : Y \to W$[3]. Since being smooth is local on source and target, we can assume $W = \operatorname{Spec} S$, $V = \operatorname{Spec} S_0$ where $S_0 = S/I$, $X = \operatorname{Spec} R_0$ where $R_0 = S_0[T_1, \dots T_n]/J$, the latter because of étaleness (and, *a fortiori*, smoothness) of the morphism $X \to V$. In this setting, $p \in \operatorname{Spec} R_0$ is a prime ideal associated to some q/J, where $q \subseteq S_0[T_1, \dots T_n]$ is prime. Since the associated ring map $S_0 \to R_0 = S_0[T_1, \dots T_n]/J$ is smooth, hence formally smooth, there are finitely many $f_1, \dots f_m \in J$ such that $(J/J^2)_q = \langle \overline{f_1}, \dots \overline{f_m} \rangle$, as a $S_0[T_1, \dots T_n]_q$-module, and $\det\left(\frac{\partial f_i}{\partial T_j}\right) \notin q$. We can, modulo moving to an affine neighbourhood, assume that $J = (f_1, \dots f_m)$.

Consider now $Q = S[T_1, \dots T_n]$. Then $S_0[T_1, \dots T_n] = Q/L$, and there is an association that sends some prime ideal $q_0 \subseteq Q$ to q, some polynomials $g_1, \dots g_m$ to $f_1, \dots f_m$, the ideal $(g_1, \dots g_m)$ to J. Fix now a neighbourhood Y of

$$p' = q/(g_1, \dots g_m) \in \operatorname{Spec} Q/(g_1, \dots g_m).$$

Since

$$Q_{p'}/(g_1, \dots g_m)_{p'}$$

is a formally smooth S-algebra, we get the desired morphism $Y \to W$. ■

[3] This is because étale is equivalent to smooth and unramified, and once we have produced $Y \to W$ such that $O \cong_V Y \times_W V$, the fibers of $O \to V$ and $Y \to W$ are isomorphic; since $O \to V$ is unramified, upon moving to a smaller Y if necessary we obtain that $Y \to W$ is unramified as well. See [Gro64b], 18.1.1.

THEOREM 2.3.6. — *If* $\varphi : V \to W$ *is a closed immersion, then*

$$\varphi : (V(k), \mathcal{E}_k(V)) \to (W(k), \mathcal{E}_k(W))$$

is a closed embedding.

Proof. Assume that V is actually a closed subvariety of W and f is the inclusion map. Then $V(k) \subseteq W(k)$ is closed in the Zariski topology and thus in the étale-open topology. Since f is continuous, the étale-open topology on $V(k)$ is finer than the subspace one; viceversa, if $U \subseteq V(k)$ is an étale image given by $g : X \to V$ and $p \in X$ is a (scheme-theoretic) point, then by the lemma there is an open neighbourhood O_p of p and an étale morphism $h_p : Y_p \to W$ such that $O_p \cong Y_p \times_W V$ as V-schemes. Let $U'_p = h_p(Y_p(k))$: since the O_ps cover X, we can extract a finite subcover $X = \bigcup_{p\in I} O_p$ and thus $U = \bigcup_{p\in I} U'_p \cap V(k))$ is a finite cover. Since $\bigcup_{p\in I} U'_p$ is an étale image, this shows that U is also open in the subspace topology. As a consequence, f is a closed (topological) embedding. ■

2.4 Systems of topologies

MANY assignments of topologies to varieties over a fixed field k can naturally be turned into functors. As a toy example, consider a k-variety V and the Zariski topology on its set of k-points, $V(k)$. This defines a functor $Z_k : (\mathbf{Var}_k) \to (\mathbf{Top})$; moreover, by definition closed and open immersions are transformed into closed and open embeddings of these topological spaces. A similar phenomenon happens with the order topology on $\mathbb{R}$, the valuation topology on any valued field or the analytic topology on $\mathbb{C}$. We distill this into a definition.

Definition 2.4.1. — A functor $T : (\mathbf{Var}_k) \to (\mathbf{Top})$ that lifts the functor of points is said to be a **system of topologies** if open (closed) immersions are sent to open (closed) embeddings. If V is k-variety and T is a system of topologies, then the associated topology on $V(k)$ is called the **T-topology** and sometimes denoted by $T(V)$ (even though technically $T(V)$ is the couple made up by $V(k)$ and its topology).

SINCE k-varieties are built up by *affine* k-varieties, the relevant information is contained in what a system of topologies does on the subcategory $(\mathbf{AffVar}_k)$ of $(\mathbf{Var}_k)$. In fact, suppose you are given a functor T from $(\mathbf{AffVar}_k)$ to $(\mathbf{Top})$ that sends closed (open) immersions to closed (open) embeddings. Then,

THEOREM 2.4.2. — *There is a unique system of topologies* $T' : (\mathbf{Var}_k) \to (\mathbf{Top})$ *such that, for every affine* k*-variety* V*, the* T*-topology on* $V(k)$ *and the* T'*-topology on* $V(k)$ *coincide.*

Proof. Pick any k-variety V and cover

$$V = \bigcup_{i \in I} V_i$$

by affine open subvarieties. Declare that $U \subseteq V(k)$ is open in the T'-topology if and only if $U \cap V_i(k)$ is open in the T-topology on V_i for every $i \in I$. This gives a topology on $V(k)$ that, a priori, depends on the choice of the cover. However, if

$$V = \bigcup_{j \in J} V'_j$$

is another cover and denote by T'' the topology on $V(k)$ defined by this cover.

We need to show that T' and T'' agree on $V(k)$: for each i, j, define

$$O_{i,j} := V_i \cap V'_j$$

so that $O_{i,j}$ is affine open in V (to see that, take the fiber product of the spectra that witness that V_i and V'_j are affine open). Then, for any $U \subseteq V(k)$, $U \cap V_i(k)$ is open for $T(V_i)$ for all i if and only if $U \cap O_{i,j}(k)$ is open for $T(O_{i,j})$ for all i, j if and only if $U \cap V'_j(k)$ is open for $T(V'_j)$ for all j, so $T'(V) = T''(V)$.

Hence, we have an assignment $T' : (\mathbf{Var}_k) \to (\mathbf{Top})$. To show that it is a system of topologies, we first need to show that it is a functor, i.e. that if $\varphi : V \to W$ is a morphism, then $\varphi : V(k) \to W(k)$ is T'-continuous. Cover V with affine open subvarieties,

$$V = \bigcup_{i \in I} V_i,$$

and similarly for W,

$$W = \bigcup_{j \in J} W_j.$$

For any $U \subseteq W(k)$ which is T'-open, we need to check that, for all $i \in I$, $\varphi^{-1}(U) \cap V_i$ is T-open. Now,

$$\varphi^{-1}(U) \cap V_i = \varphi^{-1}\left(\bigcup_{j \in J} U \cap W_j\right) \cap V_i = \bigcup_{j \in J} \varphi^{-1}(U \cap W_j) \cap V_i.$$

Since $\varphi|_{V_i}$ is a continuous map for the T-topology, it follows that this is an open subset of V_i (since $U \cap W_j$ is T-open for every j).

Suppose now that $\varphi : V \to W$ is an open immersion. Then $\varphi : V(k) \to W(k)$ is already injective, so we only need to check that it is open for the T-topology. Suppose that $U \subseteq V(k)$ is open: if

$$W = \bigcup_{i \in I} W_i$$

is an open cover by affine open subvarieties, let

$$\varphi^{-1}(W_i) = \bigcup_{j \in J} V_{ij}$$

be a cover by affine open subvarieties so that

$$\varphi(U) \cap W_i(k) = \bigcup_{j \in J} \varphi(U \cap V_{ij}(k))$$

and hence, since $\varphi|_{V_{ij}}$ is T-open, it follows that $\varphi(U) \cap W_i(k)$ is T-open. This holds for every $i \in I$, so $\varphi(U)$ is T'-open.

Finally, suppose $\varphi : V \to W$ is a closed immersion. Then again $\varphi : V(k) \to W(k)$ is injective and we need to check that it is closed. Let $C \subseteq V(k)$ be closed; we aim to show that $\varphi(C) \subseteq W(k)$ is closed.

✱ Don't let the unions scare you – all index sets are finite, here, so unions of closed sets are closed.

Let

$$W = \bigcup_{i \in I} W_i$$

be an open cover by affine subvarieties, and let $\varphi^{-1}(W_i)$ be covered by affine open subvarieties V_{ij} as before. Then $\varphi|_{V_{ij}}$ is a closed map, so $\varphi|_{V_{ij}}(C \cap V_{ij}(k))$ is closed in $W_i(k)$ for all j, in particular

$$\varphi(C) \cap W_i(k) = \bigcup_j \varphi(C \cap V_{ij})$$

is closed for every $i \in I$, and so $\varphi(C)$ is closed. ■

THE next step is considering possible *relationships* between these functors. As mentioned, there are several examples of systems of topologies – some are finer, some are coarser.

SUPPOSE you have two different systems of topologies, T_1 and T_2, and that for any k-variety V the T_1-topology on $V(k)$ is finer than the T_2-topology, i.e. $T_2(V) \subseteq T_1(V)$. We can establish a *natural transformation* between the two functors T_1 and T_2 in the following way: for each k-variety V, consider the identity map

$$\mathrm{id}_{V(k)} : (V(k), T_1(V)) \to (V(k), T_2(V))$$

that is continuous because $T_2(V) \subseteq T_1(V)$. When this happens, we write $T_2 \leq T_1$ and say **T_1 refines T_2**.

ONCE we establish this nomenclature, one can easily see that the Zariski system of topologies sits at the bottom of this ordering. Since Zariski open subsets naturally correspond to open immersions, the corresponding subsets are open subsets of the k-points of the variety.

2.4.1 *Properties of the refinement relationship*

THE relationship $\leq$ between systems of topologies can be established by looking at affine space exclusively.

THEOREM 2.4.3. — *Let T, T' be two different systems of topologies. Suppose that, for every $n \in \mathbb{N}$, $T'(\mathbb{A}^n_k) \subseteq T(\mathbb{A}^n_k)$; then $T' \leq T$.*

Proof. Since the T'-topology on $\mathbb{A}^n_k(k)$ refines the T-topology and closed immersions turn into closed embeddings, the statement is true for every affine variety; moreover, it is true for any k-variety because it is locally affine. ∎

IF you consider a system of topologies, then very often it will not assign to $V \times W$ the product of the topologies it had assigned to V and W. This happens, for example, for the Zariski system of topologies. Even for the étale-open system, this will happen only under special conditions (namely, that the system is induced by a field topology). One of the inclusions still holds.

LEMMA 2.4.4. — *Suppose T is a system of topologies. If V and W are k-varieties, then the T-topology on $V \times W$ refines the product topology.*

Proof. Since projections $\pi_1 : V \times W \to V$ and $\pi_2 : V \times W \to W$ are morphisms, they are transformed into continuous maps in the T-topology. ∎

AS a consequence, we need only check that a certain topology is discrete on $\mathbb{A}^1_k$ to deduce that the whole system is the discrete one. This will be particularly useful when we will use the étale-open topology to characterize largeness of fields.

COROLLARY 2.4.5. — *If T is a system of topologies and the T-topology on $\mathbb{A}^1_k$ is discrete, then T is the discrete system of topologies.*

2.4.2 *Restriction of systems of topologies*

WORKING with varieties often involves changing their base field, for example through Weil descent or classical base change. What follows is a way of implementing this machinery into the framework of systems of topologies.

Definition 2.4.6. — Suppose $k \subseteq K$ is a field extension and T is a system of topologies defined on K. For any k-variety V, consider its base change V_K and identify it with $V_K(K)$: then we can see $V(k)$ as a subspace of $V_K(K)$ endowed with the T-topology. The subspace topology induced on $V(k)$ will be called the **restriction** of the system of topologies T, $\mathrm{Res}_{K/k}(T)$.

WE claim that this defines a transformation on systems of topologies (or, in other words, a functor between certain categories).

PROPOSITION 2.4.7. — *If T is a system of topologies, then $R := \mathrm{Res}_{K/k}(T)$ is a system of topologies.*

Proof. Let $f : V \to W$ be a map of k-varieties. Consider its base change to $f_K : V_K \to W_K$, so the induced map $f_K : V_K(K) \to W_K(K)$ will be continuous with respect to T. Since $f = f_K|_{V(k)}$, f is also R-continuous. Suppose now that f were an open (closed) immersion. Assume V is an open (closed) subvariety of W and f is the inclusion; then $f_K : V_K \to W_K$ is still an open (closed immersion), so we can also assume V_K is an open (closed) subvariety of W_K. If U is R-open in $V(k)$, then $U = U' \cap V(k)$ for some T-open $U' \subseteq V_K(K)$ which is also T-open in $W_K(K) \cong W(K)$. Since $U \subseteq V(k)$, we can also rewrite $U = U' \cap W(k)$, and hence we get that U is R-open in $W(k)$. Similarly for closed subsets. ∎

SUPPOSE we considered the category $\mathcal{C}_k$ defined in the previous section, i.e. the poset of systems of topologies on k together with the $\leq$ relationship. Similarly, there is a category $\mathcal{C}_K$, and the restriction of systems of topologies establishes a functor

$$\mathrm{Res}_{K/k} : \mathcal{C}_K \to \mathcal{C}_k,$$

since the construction of the restriction preserves $\leq$.

AS a toy example of the effect of this functor,

PROPOSITION 2.4.8. — *Denote by Z_k and Z_K the Zariski systems of topologies over $k \subseteq K$. Then $\mathrm{Res}_{K/k}(Z_K)$ agrees with Z_k.*

Proof. We only need to work in $\mathbb{A}^n_k(k) = k^n$, since the $\leq$ relationship can be checked on affine space, and moreover we already know that $Z_k \leq \mathrm{Res}_{K/k}(Z_K)$. Hence, suppose $C \subseteq K^n$ is Zariski closed, i.e. $C = V(f_1) \cap \cdots \cap V(f_m)$ for some $f_1, \ldots f_m \in K[x_1, \ldots x_n]$. Showing the reverse inequality boils down to showing that $C \cap k^n$ is Zariski closed, i.e. finding equations with coefficients in k for $C \cap k^n$.

To do so, we can assume $m = 1$ (since intersections of closed subsets are closed). Let $f = \sum_I a_I x^I$ be the equation of C. Let $\langle a_I \rangle_k \subseteq K$ be the k-linear span of the coefficients of f, which is finite dimensional hence $\langle a_I \rangle_k = kb_1 \oplus \cdots \oplus kb_d$ for some $b_1, \ldots b_d \in K$. Then we can rewrite

$$f = \sum_I a_I x^I = \sum_{i=1}^{d} g_i(x) b_i$$

for some $g_i \in k[x_1, \ldots x_n]$. Then $C \cap k^n = V(g_1, \ldots g_d)$ is Zariski closed in k^n. ∎

2.4.3 *Restricting $\mathcal{E}_k$*

FINALLY, we look at what happens to the étale-open system of topologies when we apply this functor.

DENOTE by $\mathcal{E}_K$ the étale-open system on K. In the case of the étale-open system of topologies, algebraic extensions $k \subseteq K$ determine restrictions $\mathrm{Res}_{K/k}(\mathcal{E}_K)$ that are *refined* by $\mathcal{E}_k$.

IN general categorical nonsense, the **Weil restriction** of a K-variety X is defined to be the k-variety $\mathrm{Res}_{K/k}X$ that represents the functor from K-varieties to sets defined by $\mathrm{Res}_{K/k}X(S) := X(S \times L)$. If $X = \mathrm{Spec}\, K[x_1, \dots x_n]/(f_1, \dots f_m)$ then the Weil restriction can be computed: suppose $e_1, \dots e_r$ is a k-basis of K and set $x_i = y_{1,i}e_1 + \dots + y_{r,i}e_r$ for some new variables $y_{i,j}$. Let $g_{\ell,h}$ be the polynomials with coefficients in k and variables $y_{i,j}$ such that $f_\ell = g_{\ell,1}e_1 + \dots + g_{\ell,s}e_s$. Then $\mathrm{Res}_{K/k}X := \mathrm{Spec}\, k[y_{i,j}]/(g_{\ell,h})$. Moreover, to each map $f : V \to W$ of K-varieties comes attached a map of the restrictions, $\mathrm{Res}_{K/k}(f) : \mathrm{Res}_{K/k}(V) \to \mathrm{Res}_{K/k}(W)$[4]. We recall some properties of the Weil restriction; note that it might not exist in general, but it certainly exists for affine varieties (by the construction we have just exhibited). More generally, the Weil restriction exists for any *quasi-projective* variety (by gluing together the restrictions of an affine open covering).

PROPOSITION 2.4.9 *([Bos90], 7.6.2; [CGP15], A.5.2(4)). — Suppose $k \subseteq K$ is a finite extension, V, W are affine k-varieties (so that $\mathrm{Res}_{K/k}(V_K)$ and $\mathrm{Res}_{K/k}(W_K)$ exist). Then,*

1. *the Weil restriction and base change are adjoint functors; in particular, there is a natural morphism $V \to \mathrm{Res}_{K/k}(V_K)$, which is a closed immersion,*

2. *if $f : V \to W$ is an open immersion (closed immersion, étale map) of K-varieties, then $\mathrm{Res}_{K/k}(f) : \mathrm{Res}_{K/k}(V) \to \mathrm{Res}_{K/k}(W)$ is an open immersion (closed immersion, étale map) of k-varieties.*

THESE facts allow us to prove a technical lemma, which in turn will allow us to prove that $\mathcal{E}_k$ refines $\mathrm{Res}_{K/k}(\mathcal{E}_K)$.

LEMMA 2.4.10. — *Suppose K is a finite extension of k, V is a k-variety such that the Weil restriction of V_K exists. If $U \subseteq V_K(K)$ is an étale image in K, then $U \cap V(k)$ is an étale image in k.*

Proof. Suppose $U = h(X(K))$ for some étale morphism of K-varieties $h : X \to V_K$. Let $X = X_1 \cup \dots \cup X_n$ for some affine open X_i: then there is a natural étale morphism

$$\hat{h} : \hat{X} = X_1 \sqcup \dots \sqcup X_n \to V_K$$

which is étale and such that $U = \hat{h}(\hat{X}(K))$, so we can always assume that X is affine (replacing it with $\hat{X}$). Then $\mathrm{Res}_{K/k}X$ exists and so we can consider $\mathrm{Res}_{K/k}h$: since this is an étale morphism $\mathrm{Res}_{K/k}X \to \mathrm{Res}_{K/k}V_K$, U is a k-étale image in $\mathrm{Res}_{K/k}V_K(k)$ (identified with $V_K(K)$). Since $V \to \mathrm{Res}_{K/k}V_K$ is a closed immersion, the induced map is a closed embedding and $U \cap V(k)$ is an étale image in $V(k)$. ■

[4] In other words, $\mathrm{Res}_{K/k}$ is a functor.

WE can now prove the fact that, in algebraic extensions, the étale-open topology on the smaller field has at least the same open sets as the restriction of the étale-open topology from the bigger fields.

THEOREM 2.4.11. — *If $k \subseteq K$ is algebraic, then $\mathcal{E}_k$ refines* $\mathrm{Res}_{K/k}(\mathcal{E}_K)$.

Proof. Let V be an affine k-variety. The previous lemma takes care of the finite case. In the infinite case, suppose X is a K-variety and $g : X \to V_K$ is étale: consider an intermediate extension $k \subseteq F \subseteq K$ such that F/k is finite and X and g are defined over F, so $X = Y_K$ and $g = f_K$ for some F-variety Y and morphism of F-varieties $g : Y \to V_F$. Then

$$U = \bigcup_{F \subseteq E \subseteq K} f_E(Y_E(E)) \cap V(k)$$

where $F \subseteq E \subseteq K$ is finite. Each element of the union is $\mathcal{E}_k$-open, so U is $\mathcal{E}_k$-open. ■

IF the extension is particularly well-behaved, then we get that the two topologies actually agree.

PROPOSITION 2.4.12. — *Suppose that $k \subseteq K$ is purely inseparable, then $\mathcal{E}_k$ agrees with* $\mathrm{Res}_{K/k}(\mathcal{E}_k)$.

Proof. If $k \subseteq K$ is purely inseparable, it is in particular algebraic, hence one inclusion is proven; we need to show that $\mathrm{Res}_{K/k}(\mathcal{E}_K)$ refines $\mathcal{E}_k$. Let $U = f(X(k)) \subseteq V(k)$ be an étale image, where $f : X \to V$ is an étale morphism of k-varieties. By base changing to K we obtain another étale morphism $f_K : X_K \to V_K$ and we call $U' = f_K(X_K(K))$. On the one hand, $U \subseteq U' \cap V(k)$. On the other, let $p \in U' \cap V(k)$, so that in particular $\kappa(p) \cong k$. Consider now the fiber $f_p : X_p = X \times_V \operatorname{Spec} \kappa(p)$, which is étale – as it is the base change of an étale map. As $p \in f_K(X_K(K))$, the fiber over p contains a point q and $\kappa(q) \subseteq K$ is a separable extension of k, thus $\kappa(q) = k$ and so $q \in X(k)$. In particular, $p = f(q) \in f(X(k)) = U$. ■

2.4.4 *Encoding algebraic properties*

HIC Rhodus, hic saltus. We can finally appreciate the full power of the étale-open system of topologies as a *dictionary* between algebraic and topological properties, and more generally as a natural system of topologies generalizing topologies emerging in various contexts.

WE aim to show the following:

THEOREM 2.4.13. — *k is not separably closed if and only if the $\mathcal{E}_k$-topology on $V(k)$ is Hausdorff for all quasi-projective k-varieties V.*

FIRST of all, we only need to work over the affine line.

LEMMA 2.4.14. — *For any system of topologies T over k, the T-topology on $\mathbb{A}^1(k)$ is Hausdorff if and only if the T-topology is Hausdorff on any quasi-projective k-variety V.*

Proof. One implication is immediate. For the other, assume that V is a quasi-projective k-variety. Then, by definition, there is a morphism $V \to \mathbb{P}^n_k$ and the map on points $V(k) \to \mathbb{P}^n_k(k)$ is injective and continuous. In particular, if $\mathbb{P}^n_k(k)$ is Hausdorff then so is $V(k)$. Consider now $a, b \in \mathbb{P}^n_k(k)$ distinct points, and consider an hyperplane going through both of them, i.e. a copy of $\mathbb{A}^n_k$ inside $\mathbb{P}^n_k$ such that $a, b \in \mathbb{A}^n_k(k)$. Then, since the product of Hausdorff spaces is Hausdorff and the T-topology on $\mathbb{A}^n_k(k)$ refines the product of the T-topologies on the $\mathbb{A}^1_k(k)$s, we conclude that a, b can be separated by open subsets. In particular, that the T-topology on $\mathbb{P}^n_k(k)$ is Hausdorff. ■

LEMMA 2.4.15. — *Suppose $k \subseteq K$ is finite. If the $\mathcal{E}_K$-topology on $\mathbb{A}^1_K$ is Hausdorff, then the $\mathcal{E}_k$-topology on $\mathbb{A}^1_k$ is Hausdorff.*

Proof. This is a consequence of the fact that the étale-open topology on k refines the restriction of the one from K. ■

MOREOVER, we can already reduce the statement a bit: if k is separably closed, the étale-open topology is the Zariski topology, and so it is clearly very much not Hausdorff. We only need to show the reverse implication. Notice that by the previous lemma, we can move to a finite extension $k \subseteq K$ and exhibit a disjoint pair of non-empty $\mathcal{E}_K$-open subsets of $\mathbb{A}^1_K(K)$. To do that, we can apply the following lemma:

LEMMA 2.4.16. — *Suppose that k is not separably closed. Then there are finite field extensions $k \subseteq K \subseteq L$ such that either L is an Artin-Schreier extension of K, or there is a prime $p \neq \operatorname{char}(k)$ such that K contains a primitive pth root of unity and $L = K(a)$ for some $a \in L \setminus K$, $a^p \in K$.*

Proof. Since k is not separably closed, we have non-trivial finite Galois extensions; let $p > 1$ be minimal such that there are $k \subseteq K \subseteq L$ with L/K a Galois extension of degree p. We aim to show p is prime, so let $q \mid p$ and pick a subgroup H of $\operatorname{Gal}(L/K)$ of order q. Consider $L' = L^H$ so that L/L' is Galois of order q. By minimality, $p = q$ and it is prime. Hence if $p = \operatorname{char}(k)$, then L/K is Artin-Schreier. If $p \neq \operatorname{char}(k)$, let ζ be a primitive pth root of unity and consider $K(\zeta)$. By minimality of p, since $[K(\zeta) : K] \leq p-1$, then $\zeta \in K$ and thus L/K is a Kummer extension, i.e. $L = K(a)$ for some $a \in L \setminus K$ with $a^p \in K$. ■

WE can now prove theorem 2.4.13. By the lemma, there is a finite extension K of k such that either the pth power map $K^\times \to K^\times$ is not surjective for some prime $p \neq \operatorname{char}(k)$, or the Artin-Schreier map $K \to K$ is not surjective. In the first case, consider $P \subseteq (K^\times)(K)$ as the image of the pth power map; in the second case, consider $P \subseteq \mathbb{A}^1_K(K)$ as the image of the Artin-Schreier map. In the first case, P is a non-trivial open subgroup of $(K^\times, \cdot)$; in the second case, P is a non-trivial open subgroup of $(K, +)$. For any $a \in (K^\times)(K) \setminus P$, the set $P' = aP$ (respectively, $P' = a + P$) is open and disjoint from P, hence P and P' witness that the topology on $\mathbb{A}^1_K(K)$ is Hausdorff.[5]

MOREOVER, the étale-open topology is an usual suspect in the case of separably closed fields (in some sense, since the Zariski system of topologies is refined by every other system of topologies, this tells us that the étale-open topology on separably closed fields carries the bare minimum of information, and nothing more).

PROPOSITION 2.4.17. — *Suppose k is separably closed, then for any k-variety V the étale-open topology agrees with the Zariski topology.*

Proof. Recall that the étale-open topology refines the Zariski topology (by 2.3.1). Hence we only need to show the reverse inclusion. Let $\varphi : X \to V$ be an étale morphism of k-varieties and call $U = \varphi(X(k))$. Notice that, at the scheme-theoretic level, $\varphi(X) \subseteq V$ is an open subvariety[6]. For now, assume k is algebraically closed: then $\varphi(X)(k) = \varphi(X(k))$.

Consider now k separably closed, and $L = k^{\mathrm{alg}}$. Then $\mathcal{E}_L$ is the Zariski topology and, since $k \subseteq K$ is purely inseparable, by 2.4.12 $\mathcal{E}_k$ agrees with $\mathrm{Res}_{L/k}(\mathcal{E}_L)$. Thanks to 2.4.8, $\mathrm{Res}_{L/k}(\mathcal{E}_L)$ is exactly the Zariski system of topologies on k. ■

2.4.5 *Large fields, reprise*

WE can finally characterize large fields as precisely the class of fields over which the étale-open topology carries information.

THEOREM 2.4.18. — *k is not large if and only if $\mathcal{E}_k$ is the discrete system of topologies.*

Proof. Suppose k is not large, hence there is a smooth k-curve C such that $C(k)$ is finite and non-empty. Let $p \in C(k)$ and consider a local coordinate $f \in \mathcal{O}_{C,p}$. There is an open neighbourhood $U \subseteq C$ of p such that $f : U \to \mathbb{A}^1_k$ is étale, hence $f(U(k)) \subseteq \mathbb{A}^1_k(k)$ is a finite, open subset. This implies that the étale-open topology on $\mathbb{A}^1_k(k)$ is the discrete one and hence that $\mathcal{E}_k$ is the discrete system of topologies. Viceversa, suppose $\mathcal{E}_k$ is the discrete system of topologies. Let C be a smooth k-curve such that $C(k) \neq \emptyset$; pick some smooth point $p \in C(k)$. Since the étale-open topology on $C(k)$ is discrete, $\{p\}$ is an étale image and so there is an étale map $f : X \to C$ such that $f(X(k)) = \{p\}$. By étaleness, X is also a smooth k-curve and, since étale maps are finite-to-one, $X(k)$ is non-empty and finite. As a consequence, k is not large. ■

[5] This is enough because the action of the affine group on K is 2-transitive, i.e. the induced action on K^2 is transitive.

[6] This is because étale maps are universally open, in particular they are open. See [Gro64c], 2.4.6.

THIS allows us to derive a somewhat natural fact about large fields. Suppose $k \subseteq K$ is algebraic, and k is large. If K were not large, then $\mathcal{E}_K$ would be discrete, hence $\mathrm{Res}_{K/k}(\mathcal{E}_K)$ would be discrete. Since, by 2.4.11, $\mathcal{E}_k$ refines $\mathrm{Res}_{K/k}(\mathcal{E}_K)$, we obtain that $\mathcal{E}_k$ would also be discrete, and hence k would not be large, a contradiction. We have proven,

COROLLARY 2.4.19. — *Suppose $k \subseteq K$ is algebraic and k is large. Then K is large.*

2.5 Complements

THE next two sections will be somehow complementary to the topic, in that they will not be necessary to understand the other chapters, but somehow try to answer natural questions around them.

2.5.1 *Definability*

THE first question is: how *elementary* is the étale-open topology? or more precisely, what is the elementary content of this topology? What kind of model-theoretic tools can we use to understand it? The answers are not exactly satisfactory. In fact, it is not clear at all – and it is probably false – that the étale-open topology is somehow *seen* by the theory of the field. While it is true (see below) that étale images remain étale images under elementary extensions, the situation with open subsets in general remains elusive. Throughout this section, fix an extension of fields $k \subseteq K$ which is also elementary in the language of fields, i.e. $k \preceq K$.

THEOREM 2.5.1. — *Suppose $X = \varphi(k)$ is a definable set. If X is a k-étale image, then $X^* = \varphi(K)$ is a K-étale image.*

Proof. Suppose $X \subseteq k^n$, so that there is an étale map $f : V \to \mathbb{A}^n$ such that $X = f(V(k)) \subseteq \mathbb{A}^n(k) = k^n$. Note that we can assume that V is smooth and has dimension n. Let $g : k[x_1, \dots x_n] \to \frac{k[y_1,\dots y_m]}{(h_1,\dots h_{m-n})}$ be the induced map of k-algebras, and let $\overline{p_i} := g(x_i)$, $i = 1, \dots n$. Rewrite

$$S := \frac{k[y_1, \dots y_m]}{(h_1, \dots h_{m-n})} \cong \frac{k[y_1, \dots y_m][x_1, \dots x_n]}{(h_1, \dots h_{m-n}, x_1 - p_1, \dots x_n - p_n)}$$

and rename the generators of the ideal as $\tilde{h}_1, \dots \tilde{h}_m$. If $R = k[x_1, \dots x_n]$, then $S \cong R[y_1, \dots y_m]/(\tilde{h}_1, \dots \tilde{h}_m)$ and the map f is étale if and only if the element $J := (\partial \tilde{h}_i / \partial y_j)_{i,j}$ is a unit in S.

✱ This equivalence is explained, for example, in chapter 3, section 5 of [Mum99]. It boils down to the fact that being étale is equivalent to $\Omega_{S/R} = (0)$, where $\Omega_{S/R}$ is defined as the S-module generated by the symbols $\mathrm{d}X_1, \dots \mathrm{d}X_n$ modulo the relations $\sum_{j=1}^n \partial f_i / \partial X_j \cdot \mathrm{d}X_j = 0$. This module is then zero precisely when the jacobian is invertible.

Let θ be the sentence that says that $1 \in (\tilde{h}_1, \dots \tilde{h}_m, J) \subseteq R[y_1, \dots y_m]$: this can be written with a first-order formula in virtue of the bounds on ideals in polynomial rings, see section 5 of [Cha97]. Then $k \vDash \theta$, hence $K \vDash \theta$ and so the map $f : V \to \mathbb{A}^n$ of K-varieties is still étale, and $X^* = f(V(K))$. ■

NOTE, however, that it is not true that, given $X = \varphi(k)$, X is an étale image if and only if a certain first-order formula holds; in other words, being an étale image is not first-order: for example, $\{0\}$ is an étale image in every finite field, but not in a pseudofinite field.

2.5.2 *Pseudofinite fields*

WE now turn to another natural question, i.e. if the étale-open topology is *always* a field topology. By field topology we mean an Hausdorff topology τ on F such that inversion, addition and multiplication are τ-continuous. This allows to *induce* a system of topologies T_τ on F. We already know the answer: since we assume that field topologies are Hausdorff, it is false for example in the case of separably closed fields. There is, however, another – somehow "less trivial" – example.

FIX a pseudofinite field F, and denote by $\mathcal{E}_F$ the étale-open system of topologies on F. We now characterize the case where a system of topologies is actually induced by a field topology.

LEMMA 2.5.2. — *Suppose T is a system of topologies. Then T is induced by a field topology on F if and only if the T-topology on $\mathbb{A}^n(F)$ is the product of n copies of the T-topology on $\mathbb{A}^1(F)$.*

Proof. On the one hand, by definition if T is induced by a field topology then the product of the topologies is exactly the topology on the product. On the other hand, let τ be the T-topology on $\mathbb{A}^1(F)$. Then by definition of system of topologies, it is a field topology (because inversion and the operations are morphisms of varieties). The system induced by τ, call it T_τ, coincides with T on affine n-space for every n by hypothesis. In particular, $T_\tau = T$. ■

WITH this in mind, we can look for a contradiction to prove the following theorem.

THEOREM 2.5.3. — *If F is pseudofinite, then $\mathcal{E}_F$ is not induced by a field topology.*

Proof. Suppose $\mathrm{char}(F) \neq 2$ (otherwise, substitute every 2 for a 3 in the upcoming equations). Let

$$W = V(y - x^2) \setminus \{(0,0)\} \subseteq \mathbb{A}^2$$

and, if $\pi : \mathbb{A}^2 \to \mathbb{A}^1$ is the projection on the first coordinate, let $P = \pi(W)$. Note that P is étale-open (actually, an étale image) and, since the subtraction map is a morphism of varieties,

$$E = \{(a, b) \in F^2 \mid a - b \in P\}$$

is also étale-open.

Suppose $\mathcal{E}_F$ is induced by a field topology, then there must be étale images $U_0, U_1 \subseteq k$ such that $U_0 \times U_1 \subseteq E$. Denote by μ the measure on definable sets in F (see [Cha97]): we aim to show that $\mu(U_1) = 0$, against the fact that every étale image (over a large field) is infinite.

Fix a finite field $\mathbb{F}_q$, for $q = p^n$, and let E be a finite extension. Let $b_1, \dots b_k \in E$ be pairwise distinct elements. First, we claim that if

$$S = \{a \in E \mid a - b_i \in P,\ i = 1, \dots k\}$$

then for a sufficiently big E, $|S| < 2^{1-k}|E|$ (†). As a consequence, using the definability of μ, we have that if $b_1, \dots b_k \in F$ and

$$S' = \{a \in F \mid a - b_i \in P,\ i = 1, \dots k\},$$

then $\mu(S') < 2^{1-k}$. This implies that

$$\mu(\{a \in F \mid a - b \in P,\ \forall b \in U_0\}) = 0,$$

but this set contains U_1, so $\mu(U_1) \leq 0$, a contradiction.

To show (†), consider the quasi-affine curve C given by the equations

$$x - b_i = y_i^2,\ x - b_i \neq 0$$

for $i = 1, \dots k$. As it is absolutely irreducible, one could apply the Lang-Weil bounds to it, leading to the required estimate of $|S|$. ■

Applications: large stable fields

D'in su la vetta della torre antica,
passero solitario, alla campagna
cantando vai finché non more il giorno;
ed erra l'armonia per questa valle.
Primavera d'intorno
brilla nell'aria, e per li campi esulta,
sí ch'a mirarla intenerisce il core.

Giacomo Leopardi, *Il passero solitario*

AS a first application of the techniques developed in the last chapter, we follow [Joh+20] and prove an instance of a famous conjecture, the Stable Fields Conjecture.

THEOREM. — *Suppose k is a stable,* ***large*** *field. Then k is separably closed.*

SOMEHOW, this boils down to translating the algebraic notion of separably closed into a topological notion, which can then interact fruitfully with stability – and particularly, with generics – to derive a contradiction.

WE then need to introduce the machinery of stability theory and, in particular, of groups (and fields) definable in a stable theory. The classical reference for this is [Poi01], though some of the proofs might not look very clear at first; in that case, I suggest checking Chernikov's notes, see [Che15].

3.1 Stable groups

LET L be a language and let T be a complete theory in this language. Fix a monster model $\mathcal{M}$ of this theory. By $\models \varphi$, we will mean that the formula φ holds in the monster. We shall define stability as the absence of a certain pattern in definable sets.

Definition 3.1.1. — An L-formula $\varphi(x, y)$ is said to have the **order property** if there are sequences of elements $\{a_i\}_{i\in\omega}, \{b_i\}_{i\in\omega} \subseteq \mathcal{M}$ such that $\varphi(a_i, b_j)$ holds if and only if $i < j$. The formula φ is said to be **stable** if it does **not** have the order property. The theory T is said to be **stable** if all of its formulae are stable. A structure M in a language L is said to be **stable** if $\mathrm{Th}(M)$ is a stable theory.

THE condition in the definition can then be relaxed to a finitary condition.

LEMMA 3.1.2. — *A formula $\varphi(x,y)$ is stable if and only if, for some positive $k \in \mathbb{N}$, there are no $\{a_1,\dots a_k\} \subseteq \mathcal{M}$ and $\{b_1,\dots b_k\} \subseteq \mathcal{M}$ such that $\varphi(a_i,b_j)$ holds if and only if $i<j$.*

Proof. Suppose $\varphi(x,y)$ were unstable, then we could produce such finite sequences for every k by truncating the infinite ones. Viceversa, suppose $\varphi(x,y)$ were stable but, for every $k \geq 1$, it could be possible to produce $\{a_1,\dots a_k\}$ and $\{b_1,\dots b_k\} \subseteq \mathcal{M}$ such that $\varphi(a_i,b_j)$ holds if and only if $i<j$. Choose tuples $(x_i \mid i<\omega), (y_j \mid j<\omega)$ of variables and consider the type

$$\pi(x) = \{\varphi(x_i,x_j) \mid i \leq j\} \cup \{\neg\varphi(x_i,x_j) \mid i>j\}.$$

It is finitely satisfiable and hence, by saturation, we can produce two infinite sequences that show that $\varphi(x,y)$ has the order property. ■

FOR the sake of completeness, I now recall a series of standard results on stable theories that can be found, for example, in chapter 8 of [TZ12]. The intention, here, is providing a glimpse into the many faces of stability, without spending too much time on results that will not be *directly* related to what we aim to do. For a set A, denote by $\mathrm{Def}_\varphi(A)$ the Boolean algebra generated by instances of φ. Then $S_\varphi(A)$ is the space of ultrafilters of $\mathrm{Def}_\varphi(A)$, whose elements we call **φ-types**.

LEMMA 3.1.3. — *For a formula $\varphi(x,y)$, the following are equivalent:*

1. *there is an infinite cardinal λ such that $|S_\varphi(B)| \leq \lambda$ whenever $|B| \leq \lambda$,*
2. *$|S_\varphi(B)| \leq |B|$ for every infinite B,*
3. *$\varphi(x,y)$ is stable,*
4. *$\varphi(x,y)$ does not have the binary tree property, i.e. there is no binary tree $(b_s \mid s \in {}^{<\omega}2)$ of parameters such that, for all $\sigma \in {}^{\omega}2$, $\{\varphi^{\sigma(n)}(x, b_{\sigma|_n}) \mid n<\omega\}$ is consistent, where $\varphi^0 = \neg\varphi$ and $\varphi^1 = \varphi$.*

THERE is a fifth way of rephrasing stability, but we first need a definition.

Definition 3.1.4. — A complete type p over some set A is **definable** over some other set B if for any formula $\varphi(x,y)$ there is an $L(B)$-formula $\psi(y)$ such that, for all $a \in A$, $\varphi(x,a) \in p$ if and only if $\psi(a)$ is true. We write $\psi(y)$ as $\mathrm{d}_p x\varphi(x,y)$.

THEOREM 3.1.5. — *A formula $\varphi(x,y)$ is stable if and only if every complete φ-type $p \in S_\varphi(A)$ is definable over A.*

WE now dwelve more specifically into the (rich) theory of groups definable in (models of) stable theories. Fix a **stable group**, i.e. a group G definable[1] in $\mathcal{M}$ whose group operation is also definable. Some examples of stable groups include abelian groups and free groups (as pure groups) and groups of the form $\mathrm{GL}_n(k)$ for some algebraically closed k. Throughout this section, a formula $\varphi(x,y)$ will be called a **subgroup formula** if $\varphi(G,a) \leq G$ for every $a \in G^{|y|}$.

FIRST, we notice that definable subgroups of stable groups enjoy some form of descending chain condition.

PROPOSITION 3.1.6. — *Suppose G is stable, then for every subgroup formula $\varphi(x,y)$ there is a natural number n such that, for every $b_1,\dots b_m \in G^{|y|}$ there are $b_{i_1},\dots b_{i_n}$ such that*

$$\varphi(G,b_1)\cap\dots\cap\varphi(G,b_m) = \varphi(G,b_{i_1})\cap\dots\cap\varphi(G,b_{i_n}).$$

Proof. Suppose, on the other hand, that there is m arbitrarily big such that

$$\varphi(G,b_1)\cap\dots\varphi(G,b_m) \subsetneq \varphi(G,b_{i_1})\cap\dots\cap\varphi(G,b_{i_n})$$

for every choice of $\{b_{i_1},\dots b_{i_n}\} \subsetneq \{b_1,\dots b_m\}$. For every $i = 1,\dots m$ there is $a_i \notin \varphi(G,b_i)$ but $a_i \in \varphi(G,b_j)$ for every $j \neq i$. For every $I = \{i_1,\dots i_r\} \subseteq \{1,\dots m\}$, let $b_I = b_{i_1}\cdots b_{i_r}$. Then $\varphi(a_i,b_I)$ is true if and only if $i \notin I$. Using compactness, this shows that $\varphi(x,y)$ has the indipendence property, i.e. there exist $\{a_i\}_{i\in\omega} \subseteq G$ and $\{b_I\}_{I\subseteq\omega}$ such that $\varphi(a_i,b_I)$ is true if and only if $i \in I$. If you now fix, for example, $c_j = b_{\{0,1,\dots,j-1\}}$, then $\varphi(a_i,c_j)$ is true if and only if $i \in \{0,\dots j-1\}$ if and only if $i < j$, so that $\varphi(x,y)$ is an unstable formula. ■

COROLLARY 3.1.7 *(Baldwin-Saxl). — Suppose G is a stable group, then for every subgroup formula $\varphi(x,y)$ there is a natural number n such that for every family $\{a_i \mid i \in \lambda\} \subseteq G^{|y|}$ there are $a_{i_1},\dots a_{i_n}$ such that*

$$\bigcap_{i\in\lambda}\varphi(G,a_i) = \varphi(G,a_{i_1})\cap\dots\cap\varphi(G,a_{i_n}).$$

Proof. By the previous result, there is a natural number n such that every finite intersection of instances of $\varphi(x,y)$ is equal to the intersection of n of them. In particular, this forms a uniformly definable family (defined by n instances of $\varphi(x,y)$). Every uniformly definable family in a stable group must satisfy a chain condition, nominally that there is a natural number m such that chains have length at most m. Otherwise, suppose there was a descending chain

$$\varphi(G,b_1) \supsetneq \varphi(G,b_2) \supsetneq \dots \supsetneq \varphi(G,b_n) \supsetneq \dots$$

for some sequence of parameters $\{b_n \mid n \in \omega\}$. Then, inductively, one can

[1] Or even type-definable.

choose elements $c_i \in \varphi(G, b_i) \setminus \varphi(G, b_{i+1})$ one obtains a sequence of elements $\{c_i \mid i < \omega\}$ such that $\varphi(c_i, a_j)$ holds if and only if $i < j$, contradicting stability. There must then be a minimal element $\varphi(G, \overline{a})$, equal to the intersection $\bigcap_{i \in \lambda} \varphi(G, a_i)$. ■

✱ Note that, in truth, this result is *local*: in other words, it doesn't require the stability of *all formulae*. It is sufficient that $\varphi(x, y)$ is stable, since then all Boolean combinations of it remain stable. This will become relevant when we move on to local stability theory.

LET us make a brief stop that will not be immediately useful, because this corollary allows us to (locally) define connected components. For every formula φ, consider G^0_φ to be the intersection of all finite index subgroups of G defined by instances of φ. By the corollary, G^0_φ is the intersection of finitely many such subgroups, hence definable.

Definition 3.1.8. — The **connected component** of G is $G^0 = \bigcap_\varphi G^0_\varphi$. G is said to be **connected** if $G^0 = G$. In general, G^0 does not have finite index, but it has bounded index – in fact, $[G : G^0] \leq 2^{|T|}$.

AS an example, consider as G a saturated elementary extension of $\mathbb{Z}$. Then $G^0 = \bigcap_{n<\omega} nG$ so that $G/G^0 \cong \hat{\mathbb{Z}}$.

IN general, G^0 is not definable (it is infinitely definable). For now, let's forget about G^0, and leave the stage to the central tool we will use.

Definition 3.1.9. — A definable subset $A \subseteq G$ is **generic** if there are $a_1, \dots a_n \in G$ such that $G = a_1 A \cup \cdots \cup a_n A$. A type p is **generic** if it contains only generic formulae.

NOTICE that Poizat distinguishes between left, right and bilateral generics (and in the end, they coincide); since we will only work with abelian groups, we only need left genericity to start with. Moreover, note that G^0 (or rather, its type $G^0(x)$) is generic: its formulae define finite index subgroups, which are generic.

LEMMA 3.1.10. — *Suppose $A \subseteq G$ is definable and G is stable. If A is not generic, then $G \setminus A$ is.*

Proof. Suppose that $G \setminus A$ is not generic, then for every tuple $a_1, \dots a_n \in G$ there is an x such that $xa_i \in A$ for all i. Since A is not generic, for every $b_1, \dots b_n$ there is y such that $yb_i \notin A$ for all i. We now produce sequences $\{c_i\}_{i\in\omega}, \{d_i\}_{i\in\omega}$ such that $\psi(x, y) = xy \in A$ has the order property: start with any c_1. Given $c_1, \dots c_n$, choose d_n so that $d_n c_i \notin A$ for all $i \leq n$; given $d_1, \dots d_n$, choose c_{n+1} so that $c_{n+1} d_i \in A$ for all $i \leq n$. We have shown that $\psi(x, y)$ has the order property. ■

NOW, we turn towards proving the so-called *fundamental theorem* of stable groups, 3.1.12. To do so, we begin by noticing that G acts[2] on the set of partial type that concentrate[3] on itself as follows: if $\pi(x)$ is a partial type and $g \in G$, then

$$g \cdot \pi(x) := \{\varphi(g^{-1} \cdot x) \mid \varphi(x) \in \pi(x)\}.$$

We can then define stabilizers under this action: for a type π, let

$$\mathrm{Stab}(\pi) := \{g \in G \mid g \cdot \pi = \pi\}.$$

THE fundamental theorem will be deduced as a corollary of the following result. The proof can be found in [Pal18], Proposition 6.13(3). We sketch one part.

THEOREM 3.1.11. — *If $\mathfrak{p}$ is a global type concentrating on G, then $\mathfrak{p}$ is generic if and only if $G^0 = \mathrm{Stab}(\mathfrak{p})$.*

Proof. Suppose $\mathfrak{p}$ is a global, generic type concentrating on G. Then its stabilizer is the intersection of definable subgroups of finite index; more specifically, for any formula φ, define $\mathfrak{p}|_\varphi$ to be the corresponding complete global φ-type, and let $\mathrm{d}_\mathfrak{p} x\varphi(x, y)$ be its definition (see 3.1.5). Then define

$$\mathrm{Stab}(\mathfrak{p}, \varphi) := \{g \in G \mid \forall y(\mathrm{d}_\mathfrak{p} x\varphi(x, y) \leftrightarrow \mathrm{d}_\mathfrak{p} x\varphi(g^{-1} \cdot x, y))\},$$

so that $\mathrm{Stab}(\mathfrak{p}) := \bigcap_{\varphi(x,y)\in L} \mathrm{Stab}(\mathfrak{p}, \varphi)$. It follows that $G^0 \subseteq \mathrm{Stab}(\mathfrak{p})$. For the other inclusion, if $H \leq G$ is definable and has finite index, then $\mathfrak{p}$ must contain $x \in g \cdot H$ for some $g \in G$. By compactness, there is some $g' \in G$ such that $\mathfrak{p}$ contains $x \in g' \cdot G^0$, and so for every $h \in G$, $g \cdot \mathfrak{p}$ contains $x \in (hg') \cdot G^0$. Hence, if $h \in \mathrm{Stab}(\mathfrak{p})$, then we can find $h_1, h_2 \in G$ such that $(h_1 h) \cdot g' = h_2 \cdot g'$, so that $h \in G^0$. ■

COROLLARY 3.1.12. — *Every coset of G^0 contains a unique global complete generic type. Then, G is connected (i.e., $G = G^0$) if and only if it has a unique generic type.*

Proof. We begin by finding a complete generic type in G^0. Notice that $G^0(x)$ is a generic type, so we can extend it to a global complete generic type $\mathfrak{p}$[4]. Now, if $\mathfrak{q}$ is another generic global type extending G^0, take some $a \models \mathfrak{p}$ and some $b \models \mathfrak{q}$. Since $G^0 = \mathrm{Stab}(\mathfrak{p}) = \mathrm{Stab}(\mathfrak{q})$, then

$$a \models \mathfrak{p} \iff b = (ba)^{-1} \cdot a \models (ba)^{-1} \cdot \mathfrak{p} = \mathfrak{p},$$

[2] Most of what we will do can be done for a more general, type-definable homogeneous space (G, S).

[3] A type π concentrates on a formula φ if $\models \pi \to \varphi$.

[4] This can be done as follows: let I be the set of non-generic formulae. Then

$$G^0(x) \cup \{\neg\varphi(x, b) \mid \varphi(x, y) \in I, b \in \mathcal{M}\}$$

is consistent, so we can complete it to a global type that will again be generic.

and viceversa. So $\mathfrak{p} = \mathfrak{q}$. By translating, we obtain a unique generic type in every coset of G^0. ■

3.1.1 *Stable fields*

NOW, consider a language $L = \{+, \cdot, 0, 1\}$, a stable L-theory T and a monster model k. There are now *two* ways in which a definable subset can be generic – with respect to the *additive* group of k or with respect to the *multiplicative* group of k (i.e., $k^\times$). This gives rise to two notions of genericity.

Definition 3.1.13. — A definable subset $X \subseteq k$ is said to be **additively generic** if there is a finite $A = \{a_1, \dots a_n\} \subseteq k$ such that $k = (X + a_1) \cup \dots \cup (X + a_n)$. A definable subset $Y \subseteq k^\times$ is said to be **multiplicatively generic** if there is a finite $A = \{a_1, \dots a_n\} \subseteq k^\times$ such that $k^\times = a_1 Y \cup \dots \cup a_n Y$.

NECESSARILY, then, one has two notions of genericity for types as well.

Definition 3.1.14. — A partial (unary) type p over k is **additively generic** if each of its formulae defines an additively generic subset of k; similarly, it is **multiplicatively generic** if each of its formulae defines a multiplicatively generic subset of k.

IT turns out, however, that this book-keeping is not *really* necessary. In particular,

FACT 3.1.15. — *Consider a stable field k. Then there is a unique additively generic type p_+, a unique multiplicatively generic type $p_\times$, $p_+ = p_\times$ and a definable A is generic if and only if p_+ concentrates on A.*

Proof. The proof proceeds by showing that $(K, +)$ and $(K^\times, \cdot)$ are connected. The argument is the same as in 3.3.4. ■

3.2 Large stable fields

WE shall prove that a large stable field is separably closed. In this case, we will contradict the non-discreteness of the topology. Later on, we will contradict the stability by exhibiting an explicit unstable formula.

HERE is the philosophy of the proof: as we know, k is *not* separably closed precisely when the étale-open topology is Hausdorff on every quasi-projective k-variety. If we then assume that our field is large, stable and not separably closed, we can produce two disjoint open sets – we can even choose them from the base, so that they are definable. Now stability comes in: they cannot both be generic, since they are disjoint, and thus the *complement* of one of them must be generic. But this means that we can translate it to cover $k^\times$, and in particular, taking complements, we can find $\{0\}$ as the intersection of finitely many étale images.

THEOREM 3.2.1. — *Large stable fields are separably closed.*

Proof. Suppose k is stable and not separably closed. By 2.4.13, the étale-open topology on $\mathbb{A}^1_k(k)$ is Hausdorff. Notice that the topology is invariant under affine transformations, since they are isomorphisms of k-varieties. Pick $U_1, U_2 \subseteq \mathbb{A}^1_k(k)$ non-empty, basic disjoint open subsets. Note that images of morphisms of varieties are definable, so U_1 and U_2 are definable: if they were both generic, the unique generic type would also concentrate on their intersection and thus they would overlap, so we can assume one of them is non-generic. Let U_1 be non-generic and, if necessary, translate it so that $0 \in U_1$. If U_1 is non-generic, then $\mathbb{A}^1_k(k) \setminus U_1$ is (multiplicatively) generic, hence there are $a_1, \dots a_n \in k^\times$ such that

$$k^\times = a_1(\mathbb{A}^1_k(k) \setminus U_1) \cup \cdots \cup a_n(\mathbb{A}^1_k(k) \setminus U_1).$$

By taking the complements, we get

$$\{0\} = a_1 U_1 \cap \cdots \cap a_n U_1,$$

so by affine invariance $\{0\}$ is the intersection of finitely many open subsets, and is thus open. It follows that the topology is discrete and so k is not large. ■

THIS result can be improved slightly; in particular, we can relax the hypothesis that the theory is stable, only requiring stability for certain kinds of formulae.

3.3 Local generics

GLOBAL stability theory is not available anymore, so we have to use a local version – i.e., we have to look at a single formula, $\delta(x, y)$, and at definable sets built using instances of it. Fix a group G definable in a structure $\mathcal{M}$. Suppose $\delta(x, y)$ is a formula such that $\delta(M, b) \subseteq G$ for all $b \in \mathcal{M}$.

Definition 3.3.1. — The formula $\delta(x, y)$ is **affine invariant** if for any $a, b, c \in G$ there is $b' \in G$ such that $a\delta(G, b) + c = \delta(G, b')$.

LET $\mathrm{Def}_\delta(G)$ be the Boolean algebra of subsets of G defined by instances of δ. Similarly, $S_\delta(G)$ is the set of complete δ-types. Notice that if δ is invariant, then for any X which is δ-definable and $a \in G$, aX is still δ-definable.

Definition 3.3.2. — A subset $Y \subseteq G$ is **generic** if there are $a_1, \dots a_n \in G$ such that $a_1 Y \cup \cdots \cup a_n Y = G$. A δ-type p is **generic** if it contains only (formulae that define) generic definable sets.

THE following theorem is proven in [CPT20] as Theorem 2.3. Note that, using Baldwin-Saxl, one can already deduce that G^0_δ is defined by finitely many instances of δ and has finite index in G.

THEOREM 3.3.3. — *Suppose that $\delta(x, y)$ is stable and affine invariant. Then there is a finite index subgroup $G^0_\delta \leq G$ which is δ-definable and minimal (in* *the sense that every other finite index δ-definable subgroup of G contains it). Moreover, every left coset of G^0_δ contains a unique generic δ-type, and for every δ-definable set X and every $a \in G$ exactly one of $aG^0_\delta \cap X$ or $aG^0_\delta \setminus X$ is generic.*

IF, moreover, G is **δ-connected**, i.e. $G^0_\delta = G$, then there is a unique generic δ-type p. If $X \in \mathrm{Def}_\delta(G)$, then exactly one of X or $G \setminus X$ is generic.

NOW, restrict to definable fields. If K is a definable field in $\mathcal{M}$, then there are two ways to see it as a definable group, either $(K^\times, \cdot)$ or $(K, +)$. Thus, we will say that a type $p \in S_\delta(K)$ is **additively generic** if it is generic for $(K, +)$, and say it is **multiplicatively generic** if it is generic for $(K^\times, \cdot)$.

PROPOSITION 3.3.4. — *Suppose that K is an $\mathcal{M}$-definable field. If $\delta(x, y)$ is stable and affine invariant, then there is a unique additive generic $p_+ \in S_\delta(K)$, a unique multiplicatively generic $p_\times \in S_\delta(K)$, and they coincide ($p_+ = p_\times$).*

Proof. We first show that there is a unique additive generic, i.e. we show that $(K, +)$ is δ-connected. Suppose $H = G^0_\delta$ is the connected component of the identity: for any $b \in K^\times$, the coset $b^{-1}H$ is δ-definable and has finite index, so $H \subseteq b^{-1}H$ and so $bH \subseteq H$. It follows that H is a non-empty ideal of K, hence $H = K$. Let p_+ be the unique additive generic. If $p_\times$ is a multiplicative generic and $p_\times \neq p_+$, there is a definable $X \in \mathrm{Def}_\delta(K^\times)$ such that $p_\times$ concentrates on X but p_+ does not. Write

$$K^\times \subseteq \bigcup_{i=1}^{n} a_i X$$

and notice that each $a_i X$ is obtained from X through a definable automorphism of $(K, +)$, so they are not additively generic and so p_+ does not concentrate on $K^\times$, which is a contradiction. It follows that $p_+ = p_\times$ and it is unique. ■

3.4 Virtually large fields with stable existential formulae

SAY a field is **virtually large** if it has some finite extension which is large.

✱ By a construction of Srinivasan (see [Sri18]), not all virtually large fields are large.

WE aim to show the following:

THEOREM 3.4.1. — *If K is virtually large and every existential L_{ring}-formula is stable, then K is separably closed.*

IN order to do that, we will need to gain control on separably closed extensions; that can be achieved through a celebrated theorem of Artin and Schreier:

THEOREM 3.4.2. — *If some finite extension of K is separably closed, then K is either separably closed or real closed.*

NOW, to the proof of 3.4.1. Suppose that K is virtually large and not separably closed. Choose a large extension $K \subset L'$ of minimal degree: if L' is separably closed, then K must be real closed hence large, against minimality. Assume then that L' is not separably closed, so we obtain (by 2.4.16) a finite extension $L' \subseteq L$ such that either the p-th power map $L^\times \to L^\times$ is not surjective for some $p \neq \mathrm{char}(L)$ or the Artin-Schreier map $L \to L$ is not surjective. As L is a finite extension of the large field L', it is large. Fix P to be the image of either the p-th power map or the Artin-Schreier map; in either case, P is an étale image in L. Moreover, P is existentially definable in K and the same holds for $+_L, \times_L : K^m \times K^m \to K^m$ where m is the degree of $K \subset L$. Let

$$\varphi(x, y_1, y_2) = [y_1 \in L^\times] \ \wedge \ [x \in (y_1 \times_L P) \ +_L \ y_2]$$

which is affine invariant. Since P and its affine images are étale-open, every instance of φ is étale-open. Fix $P' \subseteq K$ such that P' is defined by φ and $P \cap P' = \varnothing$ (for example a coset of P in the groups $(L, +)$ or $(L^\times, \times)$). Suppose φ is stable. Then either P or P' is not φ-generic, for example take P. Let $P'' = P - c$ for some $c \in P$. Then $0 \in P''$ and, since P'' is also not generic, $K^\times \setminus P''$ must be so that

$$K^\times = \bigcup_{i=1}^{n} a_i (K^\times \setminus P'')$$

and thus $\{0\} = \bigcap_{i=1}^{n} a_i P''$ is étale-open, against the largeness of L.

Applications: large simple fields

Dipinte in queste rive
Son dell'umana gente
Le magnifiche sorti e progressive.
Qui mira e qui ti specchia,
Secol superbo e sciocco,
Che il calle insino allora
Dal risorto pensier segnato innanti
Abbandonasti, e volti addietro i passi,
Del ritornar ti vanti,
E proceder il chiami.

Giacomo Leopardi, *La ginestra, o fiore del deserto*

FOR a second application, we again need some model-theoretic machinery. In particular, a slightly larger class of theories, that of *simple* theories, which – despite not enjoying all the properties of stable theories – are still relatively tame. The main source for this chapter will be [PW21].

4.1 Simple groups and generics

FIX a complete L-theory T and a monster model $\mathcal{M}$. With G we will always mean a group definable (over the empty set) in $\mathcal{M}$.

Definition 4.1.1. — Let $\varphi(x, y)$ be an L-formula and $b \in A^{|y|}$ for some $A \subseteq \mathcal{M}$. We say $\varphi(x, b)$ **divides over** A if there is an A-indiscernible sequence $\{b_n \mid n \in \omega\}$, with $b_0 = b$, such that

$$\bigcap_{n\in\omega} \varphi(\mathcal{M}, b_n) = \emptyset.$$

A (partial) type p is then said to **divide over** A if some formula in p divides over A. Non-dividing gives a notion of independence, similar to the one introduced in the stable case; we say that a is **indipendent** from B over some set A, and write $a \perp_A B$, if $\text{tp}(a/B, A)$ does not divide over A. This notion does not behave *as well* as the stable case unless we assume some further condition on T:

Definition 4.1.2. — A theory T is said to be **simple** if for any set $A \subseteq \mathcal{M}$ and *complete* type p there is $A_0 \subseteq A$ such that $|A_0| \leq |T|$ and p does not divide over A_0.

IN simple theories, dividing agrees with forking, a good independence theorem is available and many more interesting properties hold. We will, however, only focus on some properties of such theories – in particular, on the fact that there is a good notion of *genericity* available. *From now on, assume T is simple.*

Definition 4.1.3. — A definable subset $X \subseteq G$ is **f-generic** if, for every $g \in G$, gX does not divide over $\emptyset$. A type p concentrated on G (i.e., p implies the formula defining G) is **f-generic** if every formula in p is f-generic.

BEING f-generic is vaguely related to what being generic meant in the previous chapter. More precisely,

Definition 4.1.4. — A complete type $p(x) \in S(A)$ that concentrates on G is said to be **generic** if for any $g \in G$ and any realization a of p that is independent from g over A, we have $ga \perp_{\emptyset} A, g$. A partial type is **generic** if it is contained in some complete generic type.

WE can, as in the stable case, define an action of G on its partial types by

$$g \cdot \pi(x) := \{\varphi(g^{-1} \cdot x) \mid \varphi(x) \in \pi(x)\}.$$

The following fact relates the two notions of genericity; it is Proposition 3.10 in [Pil98].

THEOREM 4.1.5. — *For a partial type $\pi(x)$ over A concentrating on G, the following are equivalent:*

1. *π is generic,*
2. *π is f-generic,*
3. *for every $g \in G$, $g \cdot \pi$ does not divide over A.*

IN particular,

PROPOSITION 4.1.6. — *Suppose X is not f-generic, then there are $g_1, \ldots g_k$ in G such that $\bigcap_{j=1}^{k} g_j X = \emptyset$.*

Proof. Let A be a set of parameters over which $X = \varphi(\mathcal{M}, b)$ is defined. By 4.1.5, if X is not f-generic, then there is a translate gX, $g \in G$, which divides over A. In particular, there are finitely many $b_1, \ldots b_n$ such that $b_i \equiv_A b$ for every i and $\bigcap_{i=1}^{n} g\varphi(\mathcal{M}, b_i) = \emptyset$. But since A is a set over which X is defined, and $b_i \equiv_A b$ – or equivalently, $g\varphi(\mathcal{M}, b_i)$ is obtained from gX with an automorphism fixing A – then $g\varphi(\mathcal{M}, b_i) = gg_i X$ for some $g_i \in G$. Hence, upon renaming $g_i' = gg_i$, we obtain the necessary result. ■

PROPOSITION 4.1.7. — *An A-definable set X is f-generic if and only if it is contained in an f-generic complete type over A.*

Proof. One direction follows directly from the definition of f-generic type. On the other hand, suppose X is f-generic. Then, again by 4.1.5, this is equivalent to saying that X is contained in some complete generic type $p(x)$ which can be assumed[1] to be over A. ■

NEXT, we consider a field K definable in $\mathcal{M}$. Similarly to what happened in the stable case, there are several ways something can be *f-generic*, depending on the structure you consider. We will say that $X \subseteq K$ is **additively (multiplicatively) f-generic** if X ($X \cap K^\times$) is f-generic for $(K,+)$ (for $(K^\times,\cdot)$). We already know examples of f-generic sets: étale-open subsets. To see that, we first need to recall some technical facts, the proofs of which can be found in [PW21].

FIRST, as a hint towards the usefulness of f-generics, we again don't need to keep in mind which reduct of the original structure we are considering.

PROPOSITION 4.1.8. — *Let $X \subseteq K$ be a definable set over A. Then X is an additive f-generic if and only if it is a multiplicative f-generic. If $a \in k$ and $p(x) = \mathrm{tp}(a/A)$, then p is an additive f-generic if and only if it is a multiplicative f-generic.*

AND similarly for tuples.

PROPOSITION 4.1.9. — *Let $a \in K^n$. Then $p(x) = \mathrm{tp}(a/A)$ is an f-generic type of $(K^n,+)$ if and only if it is an f-generic type of $((K^\times)^n,\cdot)$. In particular, X is f-generic for K^n if and only if $X \cap (K^\times)^n$ is f-generic for $(K^\times)^n$.*

NOW, as a preliminary step, we show that étale images – and so étale-open sets – are somehow *generic* in the sense of algebraic geometry. In other words, they are Zariski dense.

PROPOSITION 4.1.10. — *Suppose K is large, and $X \subseteq K^n$ is a non-empty étale image. Then X is Zariski dense.*

Proof. Suppose not, i.e. $X \subseteq V(K)$ for some proper subvariety $V \subsetneq \mathbb{A}^n$. In particular, $\dim V < n$. On the other hand, if $f : W \to \mathbb{A}^n$ is the étale map defining X, then W has dimension n and $f^{-1}(V)$ is a closed subvariety of W containing $W(K)$. As K is large, $W(K)$ is Zariski dense, hence $\dim f^{-1}(V) = n$; however, étale maps are finite-to-one, so $\dim f^{-1}(V) = \dim V < n$, a contradiction. ■

COROLLARY 4.1.11. — *Suppose K is large and $\emptyset \neq X \subseteq K^n$ is étale-open. Then X is Zariski dense.*

INTUITIVELY, Zariski dense sets should be *big*; and big sets should be f-generic – think of it this way: if a set is *not* f-generic, then it is small enough that you can translate it around and obtain several disjoint copies of it. This intuition holds up to the scrutiny:

[1] See Lemma 3.3 in [Pil98].

THEOREM 4.1.12. — *Suppose K is large. Let $X \subseteq K^n$ be a definable, étale-open subset. Then X is f-generic.*

Proof. Suppose not. Then $X' = X \cap (K^\times)^n$ is not f-generic. We can assume $\bar{0} \in X$, since étale-open subsets and f-generic subsets are affine invariant, and moreover we should have $g_1 X' \cap \cdots \cap g_k X' = \emptyset$ for some $g_1, \ldots g_k \in G$. In particular,

$$Y := g_1 X \cap \cdots \cap g_k X \subseteq K^n \setminus (K^\times)^n,$$

so Y is not Zariski dense. However, since $\bar{0} \in Y$, it is non-empty and étale-open, and hence it should be Zariski dense. ■

4.2 Bounded fields

Definition 4.2.1. — A field K is said to be **bounded** if, for every $n < \omega$, there are only finitely many extensions of K of degree n inside K^{sep}.

FOR some $a = (a_0, \ldots a_n) \in K^{n+1}$, let

$$p_a(x) = a_0 + a_1 x + \cdots + a_n x^n \in K[x].$$

Let

$$\Omega_n := \{a \in K^{n+1} \mid p_a \text{ is separable and irreducible}\}.$$

It is a definable subset of K^{n+1}, on which we establish an equivalence relationship:

$$a \sim_n b \iff K/(p_a) \cong K/(p_b).$$

Through $\sim_n$, we are encoding the information on separable extensions of K in definable subsets of K^{n+1}. In particular, as a consequence of the primitive element theorem,

PROPOSITION 4.2.2. — *K is bounded if and only if, for every n, there are only finitely many $\sim_n$-classes.*

NOTE, moreover, that each $\sim_n$-class is definable in K, since $K/(p_a)$ is uniformly interpretable in K. This allows us to combine topological techniques and tools from simplicity to assess how *big* these classes are, and thus in particular how many of them there are.

4.3 Large simple groups are bounded

WE aim to show that, if K is large and simple, there are only finitely many $\sim_n$-classes for each $n < \omega$. This is where simplicity plays its role.

PROPOSITION 4.3.1. — *Suppose T is simple, X is a definable set in K and $\sim$ is a definable equivalence relation on X such that each class is f-generic. Then there are only finitely many $\sim$-classes.*

Proof. Suppose not. Let c be a tuple of parameters over which X and $\sim$ can be defined. Since $X/\sim$ is infinite, there is a class E and a canonical parameter for E, $e \notin \operatorname{acl}(c)$. Let $\theta(x, e, c)$ define E. Choose a sequence $\{e_i \mid i < \omega\}$, starting in e, such that $\{\theta(x, e_i, c) \mid i < \omega\}$ are pairwise inconsistent, and extract a c-indiscernible sequence $\{\bar{e}_i \mid i < \omega\}$ of realizations of $\operatorname{tp}(e/c)$. Then $\{(c, \bar{e}_i) \mid i < \omega\}$ is an $\emptyset$-indiscernible sequence that witnesses that $\theta(x, e, c)$ divides. ■

WE now focus on the set Ω_n together with the relation $\sim_n$. If we can show that $\sim_n$-classes are f-generic, then we are done; but now we have a tool at our disposal, or rather, we have an example of f-generic sets: étale-open sets. It can't hurt to try.

PROPOSITION 4.3.2. — *Suppose K is large. Then every $\sim_n$-class is étale-open.*

Proof. Fix $a \in \Omega_n$ and let α be a root of $p_a(x)$ in K^{sep}. Choose some variables $(x_0, \dots x_{n-1})$ and let

$$\beta(\bar{x}) = x_0 + \alpha x_1 + \cdots + x_{n-1}\alpha^{n-1}.$$

Similarly, denote by $\alpha_1, \dots \alpha_n$ the K-conjugates of α and let

$$\beta_i(\bar{x}) = x_0 + \alpha_i x_1 + \cdots + x_{n-1}\alpha_i^{n-1}.$$

Now consider

$$\Phi := \{b \in K^n \mid K(\beta(b)) = K(\alpha)\},$$

so that $b \in \Phi$ if and only if there is a $c \sim_n a$ such that $p_c(\beta(b)) = 0$. Since, moreover, $\beta(b) \in K(\alpha)$ for any $b \in K^n$, we have that $b \in \Phi$ if and only if $1, \beta(b), \dots \beta(b)^{n-1}$ are K-linearly independent. In particular, Φ is Zariski open.

We now aim to produce a non-empty étale image around a in its $\sim_n$-class. Denote by $e_1, \dots e_n \in \mathbb{Z}[\bar{x}]$ the symmetric polynomials defined by

$$e_k(\bar{x}) = \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k},$$

and let

$$G(b) := (-e_1(\beta_1(b), \dots \beta_n(b)), e_2(\beta_1(b), \dots \beta_n(b)), \dots (-1)^n e_n(\beta_1(b), \dots \beta_n(b)))$$

for any $b \in K^n$. Note that we can rewrite G as a polynomial function $G = (G_1, \dots G_n)$ where $G_i \in K[\bar{x}]$ and if $b \in \Phi$, then $G(b) \sim_n a$. This is because $p_{G(b)}$ has exactly $\beta_1(b), \dots \beta_n(b)$ as roots.

Now,

$$G(0, 1, 0, \dots 0) = (-e_1(\alpha_1, \dots \alpha_n), \dots (-1)^n e_n(\alpha_1, \dots \alpha_n)) = a.$$

Moreover, we show that the Jacobian of G at $(0,1,0,\dots 0)$ is invertible – define the following maps:

$$\begin{aligned}F(b) &= (b_0 + b_1\alpha_1 + \cdots + b_{n-1}\alpha_1^{n-1}, \dots b_0 + b_1\alpha_n + \cdots + b_{n-1}\alpha_n^{n-1}),\\ E(b) &= (e_1(b), \dots e_n(b)),\\ D(b) &= (-b_0, b_1, -b_2, \dots (-1)^n b_{n-1}).\end{aligned}$$

For any $b \in V$, we have $G(b) = D(E(F(b)))$. This allows us to compute the Jacobian of G: if we denote by $p = (0,1,0,\dots 0)$, then by the chain rule

$$\mathrm{Jac}_G(p) = \mathrm{Jac}_D(E(F(p)))\,\mathrm{Jac}_E(F(p))\,\mathrm{Jac}_F(p).$$

However, both D and F are linear in b, so their Jacobians are constant. On the one hand, $|\mathrm{Jac}_D|$ is either 1 or -1 (depending on n); on the other hand, Jac_F is a Vandermonde matrix in $\alpha_1, \dots \alpha_n$, which are all distinct, and so it has non-zero determinant. It remains to show that $\mathrm{Jac}_E(F(p))$ is invertible, where $F(p) = (\alpha_1, \dots \alpha_n)$; however, $|\mathrm{Jac}_E(F(p))|$ is also a Vandermonde determinant of the form $\prod_{1 \le i < j \le n}(\alpha_i - \alpha_j)$ (see [LP02]), so it is non-zero as long as $\alpha_1, \dots \alpha_n$ are distinct.

Thus, the open subvariety U of $\mathbb{A}^n$ given by $|\mathrm{Jac}_G(\bar{x})| \neq 0$ contains p, and $G : U \to \mathbb{A}^n$ gives an étale morphism. Since $U(K) \cap \Phi \neq \emptyset$ is Zariski open in K^n, we have an open subvariety W such that $W(K) = U(K) \cap \Phi$ and $G|_W$ is still étale. Let $X = G(W(K))$: then $a \in X \subseteq [a]_{\sim_n}$. Since a is arbitrary, it follows that $[a]_{\sim_n}$ is étale-open. ■

WE have now proven that we have a definable equivalence relation, $\sim_n$, whose classes are étale-open, thus f-generic. Under simplicity, this implies that there are only finitely many of them.

COROLLARY 4.3.3. — *If K is a large field, definable in a simple theory, then K is bounded.*

Éz valued fields

Volo ancora, ma nelle tregue del sonno
il piede non più leggero
scivola via, una mano si aggrappa
alla grondaia che scappa
vorrei volando volare
e riempire di allegrie
le spine del buio.

Dacia Maraini, *Ho sognato di volare*

WE go back to our original question. We now have a notion of *topology* available on every k-variety over any field k. This topology in particular only makes sense over large fields, so we restrict to this class. We can finally isolate the fields whose definable sets – in the language of fields – admit a topological description. We follow [WY21].

5.1 Éz fields

ÉZ fields arise as a *topological* form of model completeness for perfect fields. If k is a field, recall the étale-open topology $\mathcal{E}_k$ on k, as introduced in the previous chapters.

Definition 5.1.1. — A definable subset $X \subseteq k^n$ is **éz** if it can be written as a finite union of definable étale-open subsets of Zariski closed subsets of k^n.

Under the assumption of perfection (and largeness) of k, it is proven as Theorem C in [WY21] that if X is éz, i.e. $X = X_1 \cup \cdots \cup X_\ell$ for some definable étale-open subsets $X_i \subseteq V_i(k)$ of closed subvarieties $V_i \subseteq \mathbb{A}^n$, then $V_1, \ldots V_\ell$ can actually be chosen to be *pairwise disjoint*. As a consequence, if X is definable and decomposed in étale-open subsets $X_i \subseteq V_i(k)$ such that $V_i(k) \cap V_j(k) = \emptyset$ for $i \neq j$, then each X_i is already definable, so this request can be dropped from the definition.[1]

[1] It is, in fact, what will be done in 5.3.

Definition 5.1.2. — A field k is **éz** if it is large and every definable subset is éz.

QUANTIFIER free definable subsets of k^n are constructible, and Zariski open sets are étale-open. This means that algebraically closed fields are éz; but we can't go much farther, since quantifier elimination in the ring language is equivalent to being algebraically closed (or finite) for any division ring[2]. A general strategy to obtain éz fields is then to isolate fields with quantifier elimination in some expanded language and turn to topological considerations on the sets defined by this expanded language. This is the strategy deployed in 5.3.

IN general, éz fields are not necessarily tame (for example, $\mathbb{Q}((t))$ is éz, despite inheriting some of the awful properties of $\mathbb{Q}$), however they admit a good *geometric* dimension theory (see Theorem E in [WY21], and [Dri] for generalities on dimension in algebraically bounded fields).

THEOREM 5.1.3. — *Suppose k is éz. Then k is algebraically bounded*[3].

WE now look at certain henselian valued fields that admit quantifier elimination in a multi-sorted language. We expect them to end up being éz. As an aside, recall that henselian valued fields are large, so one of the requests of the definition is already accounted for.

5.2 The étale-open topology over henselian valued fields

BEFORE getting into the specific examples, we will need to know what happens to the étale-open topology when considered over an henselian valued field.

RECALL that a V-topology on k (see Appendix B) is said to be **t-henselian** if for any n there is an open neighbourhood U of 0 such that, if $a_0, \dots a_n \in U$, then $x^{n+2} + x^{n+1} + a_n x^n + \cdots + a_1 x + a_0$ has a root in k. Any valuation topology coming from an henselian valuation is t-henselian.

THEOREM 5.2.1. — *Suppose that k is not separably closed and admits a t-henselian topology; then the étale-open system of topologies is induced by the t-henselian topology.*

✱ The system $\mathcal{E}_k$ being induced by a t-henselian topology is actually *equivalent* to k being not separably closed and t-henselian, see Theorem B in [Joh+20].

Proof. We show it in several steps.

[2] See [Ros78] for the proof of the equivalence.

[3] A field k is said to be algebraically bounded if for every definable set $X \subseteq k^m \times k$ there are polynomials $f_1, \dots f_\ell \in k[x_1, \dots x_m, t]$ such that if the fiber $X_a = \{b \in k \mid (a,b) \in X\}$ over some $a \in k^m$ is finite, then $X_a \subseteq \{b \in k \mid f_i(a,b) = 0\}$ for some $i = 1, \dots \ell$ such that $f_i(a,t)$ is not constantly zero.

Step one: if τ is the t-henselian topology on k, and $f: V \to W$ is an étale maps of varieties, then f is τ-open.

Recall that t-henselianity implies the implicit function theorem B.2.13. In particular, if we have a polynomial map $f: k^{n+m} \to k^m$ which has a zero (a,b) such that $|\frac{\partial f_i}{\partial y_j}(a,b)_{i,j\leq m}| \neq 0$, then there are τ-open neighbourhoods of 0, $U \subseteq k^n$ and $V \subseteq k^m$, such that for all $a' \in a+U$ there is a unique $b' \in V$ such that $f(a',b') = 0$, and moreover the map $k^m \to a+U$ given by $a' \mapsto b'$ is continuous. Moreover, from the discussion around 2.2.1 we know that if $f: V \to W$ is étale, we can – locally around a k-point – assume that it is a "projection"

$$f: V(g_1, \dots g_h, f_1, \dots f_\ell) \subseteq \mathbb{A}^{n+\ell} \to V(g_1, \dots g_h) \subseteq \mathbb{A}^{n}[4],$$

with the Jacobian of $f_1, \dots f_\ell$ which is invertible at the point. Then by the implicit function theorem we can find a τ-open set $U \subseteq k^n$ and a τ-continuous function $h: U \to k^\ell$ such that $f(x, h(x)) = 0$ for every $x \in U$. In particular, $f|_{(U\times k^\ell)\cap V(g_1,\dots g_h, f_1,\dots f_\ell)}$ maps $(U \times k^\ell) \cap V(g_1, \dots g_h, f_1, \dots f_\ell)$ τ-homeomorphically[5] to $U \cap V(g_1, \dots g_\ell)$. It follows that f is a τ-open map.

As a consequence of this, τ refines the étale-open topology on any k-variety. We need to show the other refinement.

Step two: suppose τ_0 is an affine invariant topology on k and τ_1 is a non-discrete field topology on k. If some non-empty $X \subseteq k$ is τ_0-open and τ_1-bounded, then $\tau_1 \subseteq \tau_0$.

In fact, supposing $0 \in X$, if $U \in \tau_1$ is non-empty, for every $b \in U$ we can find $a_b \in k^\times$ such that $a_b X \subseteq U - b$. By affine invariance, $a_b X + b \subseteq U$ is τ_0-open, hence $U = \bigcup_{b\in U}(a_b X + b)$ is τ_0-open.

Step three: If $i: \mathbb{A}^1 \to \mathbb{P}^1$ is the open immersion $x \mapsto (x:1)$, and τ is a V-topology on k, then $X \subseteq \mathbb{A}^1(k)$ is τ-bounded if and only if $(1:0)$ is not in the closure of $i(X)$.

In fact, X is τ-bounded if and only if $0 \notin \overline{(X \setminus \{0\})^{-1}}$ if and only if $(1:0) \notin \overline{i(X)}$.

Step four: suppose that S is a system of topologies on k and τ is a V-topology on k. Suppose some non-empty S-open $U \subseteq \mathbb{A}^1(k)$ is not τ-dense in $\mathbb{A}^1(k)$, then S refines the system induced by τ.

It is enough to work on $\mathbb{A}^1$: because of 2.4.11, we need only work over affine space, and since τ is a field topology, we only care about dimension 1. We need to produce, by step three, a subset which is S-open and τ-bounded. Suppose $a \notin \overline{U}$ and let $g: \mathbb{P}^1(k) \to \mathbb{P}^1(k)$ be the fractional transformation $x \mapsto \frac{1}{x-a}$. In particular, $f(U)$ is still S-open and $f(a) = \infty \notin \overline{f(U)}$. Then $f(U)$ is τ-bounded, hence we finish.

Step five: suppose τ is a V-topology on k and there is $f \in k[x]$ such that $f' \not\equiv 0$

[4]More precisely, that it is induced by the quotient map from $\operatorname{Spec} k[x_1, \dots x_n]/(g_1, \dots g_h)$ to $\operatorname{Spec} k[x_1, \dots x_{n+\ell}]/(g_1, \dots g_h, f_1, \dots f_\ell)$.

[5]The inverse is precisely $x \mapsto (x, h(x))$.

and $f(\mathbb{A}^1(k))$ is not τ-dense in $\mathbb{A}^1(k)$. Then the τ is refined by the étale-open topology.

Let $U = D(f')$: then $f : U \to \mathbb{A}^1$ is étale and $f(U(k))$ is not dense in $\mathbb{A}^1(k)$, so we finish by step four.

Step six: we need to exhibit such an f.

Since $k \neq k^{\mathrm{sep}}$, there is a separable polynomial $f \in k[x]$ with no zeroes in k. By [ZP78], theorem 7.5, if f has no zeroes in k, then 0 is not a τ-limit point for $f(\mathbb{A}^1(k))$, so $f(\mathbb{A}^1(k))$ is not τ-dense, and hence we conclude that $\mathcal{E}_k$ refines τ on every k-variety. ■

THIS shows, in particular, that in the case of henselian valued fields, the valuation topology and the étale-open topology coincide.

5.3 The RV-language

FIX a valued field (k, v) and call R its residue field and V its value group. We aim to achieve elimination of the field quantifier through a language that, somehow, encodes the "relevant" information of elements of a valued field. Throughout this section, keep in mind the following example: for any field R and value group V, build the field of Hahn series $k = R((t^V))$. Elements of k can explicitly be written as formal power series in t, with coefficients in R and exponents in V, whose support is well-ordered. The most intuitive example is when $V = \mathbb{Z}$, so that we write $k = R((t))$: if you think of these power series as Taylor expansions of functions at a point, then the leading terms structure, as defined below, is essentially identifying "functions" up to higher[6] orders of their expansion.

Definition 5.3.1. — Let $\delta \geq 0$ be an element of the value group V. We denote by

$$\mathrm{RV}_\delta := k^\times / (1 + \mathfrak{m}_\delta) \cup \{\infty\},$$

together with its quotient map $\mathrm{rv}_\delta : k \to \mathrm{RV}_\delta$, where $\mathrm{rv}_\delta(0) = \infty$, the **leading terms structure of order δ**.

As an aside, we can induce an addition structure on RV_δ. In the $\delta = 0$ case, this is achieved by noting the following: since $1 + \mathfrak{m} \subseteq \ker(v)$, then v factors through RV_0 and thus we can look at $\ker(v) \subseteq \mathrm{RV}_0$. The former is $\mathcal{O}^\times / (1 + \mathfrak{m}) \cong R^\times$ (as a group), and hence we obtain a short exact sequence

$$1 \to R^\times \to \mathrm{RV}_0 \to V \to 0,$$

where the maps are the embedding $R^\times \to \mathrm{RV}_0$ and the valuation $\mathrm{RV}_0 \to V$. This allows us to "push" the additive structure from $R^\times$ to RV_0; we however only

[6] Where "higher" can be defined using the valuation.

obtain a *relation* $\oplus_0$ on RV_0^3, which fails to be a functional relation in many cases. More generally,

Definition 5.3.2. — Let $\delta \geq 0$. Then we say $\oplus_\delta(a, b, c)$ if and only if

$$\exists x, y, z\, [\mathrm{rv}_\delta(x) = a \wedge \mathrm{rv}_\delta(y) = b \wedge \mathrm{rv}_\delta(z) = c \wedge x + y = z].$$

MOREOVER, if $\gamma \leq \delta$ then $1 + \mathfrak{m}_\delta \subseteq 1 + \mathfrak{m}_\gamma$ and thus we have a natural map $\mathrm{rv}_{\delta,\gamma} : \mathrm{RV}_\delta \to \mathrm{RV}_\gamma$.

FROM now on, whenever we say "the RV-language" we shall mean a multi-sorted language $L_\Delta = (k, \mathrm{RV}_\delta \mid \delta \in \Delta)$, for a suitable choice of $\Delta \subseteq [0, \infty) \subseteq V$, made up by

1. the ring language on the first sort,
2. the group language together with the relation $\oplus_\delta$, for each $\delta \in \Delta$, on the sort RV_δ,
3. the function $\mathrm{rv}_\delta : k \to \mathrm{RV}_\delta$ for each $\delta \in \Delta$,
4. the function $\mathrm{rv}_{\delta,\gamma} : \mathrm{RV}_\delta \to \mathrm{RV}_\gamma$ for each $\delta \geq \gamma \in \Delta$.

WE are now ready to prove that eliminating the field quantifier in the RV-language implies being éz.

THEOREM 5.3.3. — *Suppose that* (k, v) *is a perfect*[7] *valued field that admits elimination of the field quantifier in the* RV*-language. Then* k *is éz.*

✱ The $\delta_1, \dots \delta_\ell$ come from Δ, for a suitable choice of Δ. For example, in the henselian case, with characteristic zero, then one can choose $\delta_1 = \cdots = \delta_\ell = 0$.

NOTE that, from the elimination of the field quantifier, it follows that if $X \subseteq k^n$ is definable, then there are a definable

$$E \subseteq \mathrm{RV}_{\delta_1} \times \dots \mathrm{RV}_{\delta_\ell}$$

and polynomials $f_1, \dots f_\ell \in k[x_1, \dots x_n]$ such that

$$X = \{\bar{x} \in k^n \mid (\mathrm{rv}_{\delta_1}(f_1(\bar{x})), \dots \mathrm{rv}_{\delta_\ell}(f_\ell(\bar{x}))) \in E\}.$$

WE shall show that, for any such definable E, the set

$$U = \{\bar{x} \in k^n \mid (\mathrm{rv}_{\delta_1}(f_1(\bar{x})), \dots \mathrm{rv}_{\delta_\ell}(f_\ell(\bar{x}))) \in E\}$$

is éz, and thus every definable set will be éz. Recall that, due to Theorem C in [WY21], it is enough to show that definable sets decompose as finite unions of

[7]Meaning that the base field is perfect.

étale-open (hence, in this case, valuation open) subsets of Zariski closed sets: under the perfection hypothesis, there is no need to worry about definability of the components of the union.

TO see that, we first look at the $\ell = 1$ case. Fix $\delta \geq 0$, $E \subseteq \mathrm{RV}_\delta$ and $f \in k[x_1, \dots x_n]$ and define U as above. Then the set

$$U_0 = \{z \in k^\times \mid \mathrm{rv}_\delta(z) \in E\}$$

is valuation open, since

$$U_0 = \bigcup_{z \in U_0} B_{v(z)+\delta}(z).$$

We can now reconstruct U from U_0 using the function $\Phi : k^n \setminus V(f) \to k^\times$ defined by $\Phi(\bar{x}) = f(\bar{x})$. Since Φ is valuation continuous, we have that $U = V(f) \cup \Phi^{-1}(U_0)$ is the union of a Zariski closed set and a valuation open (hence étale-open) set. In particular, U is éz.

WE now turn to the higher “dimensional” cases. Before attacking the general case, it might be instructive to look at the $\ell = 2$ case. Fix $\delta_1, \delta_2 \geq 0$, $f_1, f_2 \in k[x_1, \dots x_n]$ and $E \subseteq \mathrm{RV}_{\delta_1} \times \mathrm{RV}_{\delta_2}$. Define U as above, and assume $(\infty, \infty) \in E$. If $\bar{x} \in U$, then we have four cases:

1. $\bar{x} \in V(f_1, f_2)$,
2. $\bar{x} \in V(f_1)$ but $\bar{x} \notin V(f_2)$: then $\bar{x}$ belongs to the set
$$\bigcup_{(\infty,\lambda)\in E,\ \lambda \neq \infty} \{\bar{x} \in k^n \mid \mathrm{rv}_{\delta_2}(f_2(\bar{x})) = \lambda\},$$
3. $\bar{x} \in V(f_2)$ but $\bar{x} \notin V(f_1)$: then $\bar{x}$ belongs to the set
$$\bigcup_{(\lambda,\infty)\in E,\ \lambda \neq \infty} \{\bar{x} \in k^n \mid \mathrm{rv}_{\delta_1}(f_1(\bar{x})) = \lambda\},$$
4. $\bar{x}$ is not in $V(f_1)$ and $V(f_2)$, then $\bar{x}$ belongs to the set
$$\bigcup_{(\lambda,\gamma)\in E,\ \lambda,\gamma \neq \infty} \{\bar{x} \in k^n \mid \mathrm{rv}_{\delta_1}(f_1(\bar{x})) = \lambda\} \cap \{\bar{x} \in k^n \mid \mathrm{rv}_{\delta_1}(f_1(\bar{x})) = \gamma\}.$$

In particular, U decomposes as the union of

1. $V(f_1, f_2)$,
2. $V(f_1) \cap D(f_2) \cap \bigcup_{(\infty,\lambda)\in E,\ \lambda\neq\infty} \{\bar{x} \in k^n \mid \mathrm{rv}_{\delta_2}(f_2(\bar{x})) = \lambda\}$,
3. $V(f_2) \cap D(f_1) \cap \bigcup_{(\lambda,\infty)\in E,\ \lambda\neq\infty} \{\bar{x} \in k^n \mid \mathrm{rv}_{\delta_1}(f_1(\bar{x})) = \lambda\}$,
4. $D(f_1) \cap D(f_2) \cap \bigcup_{(\lambda,\gamma)\in E,\ \lambda,\gamma\neq\infty} [\{\bar{x} \in k^n \mid \mathrm{rv}_{\delta_1}(f_1(\bar{x})) = \lambda\} \cap$
$\cap \{\bar{x} \in k^n \mid \mathrm{rv}_{\delta_1}(f_1(\bar{x})) = \gamma\}]$,

where 1 is Zariski closed, 2 – 3 are étale-open subsets of the Zariski closed sets $V(f_1)$ and $V(f_2)$ and 4 is étale-open in k^n. It follows that U is éz.

FOLLOWING this principle, we decompose U when $\ell \geq 3$. We have

1. $V(f_1, \dots f_\ell)$,
2. $D(f_1) \cap \cdots \cap D(f_\ell) \cap \bigcup_{\bar{\gamma} \in E, \forall i\, \gamma_i \neq \infty} \bigcap_{i=1}^{\ell} \{\bar{x} \in k^n \mid \mathrm{rv}_{\delta_i}(f_i(\bar{x})) = \gamma_i\}$,
3. for any choice of $I \in 2^\ell$,

$$V(f_i \mid i \in I) \cap D(f_i \mid i \notin I) \cap \\ \cap \bigcup_{\bar{\gamma} \in E, \forall i \notin I, \gamma_i \neq \infty} \bigcap_{i \notin I} \{\bar{x} \in k^n \mid \mathrm{rv}_{\delta_i}(f_i(\bar{x})) = \gamma_i\}.$$

Note that some of these unions might be empty, depending on whether ∞ appears as a coordinate in E or not. Each of these sets is éz, and hence U is éz. Again, if $(\infty, \dots \infty) \notin E$ we ought to omit 1.

5.4 Some examples

WE survey some examples of valued fields to which the result of the previous section applies. First of all, the characteristic zero case: this is [Fle09], Proposition 4.3.

PROPOSITION 5.4.1. — *Suppose* (k, v) *is henselian,* $\mathrm{char}(k) = 0$*. Then* k *admits elimination of the field quantifier in the* RV*-language (where* $\Delta = \{0\}$ *if* $\mathrm{char}(R) = 0$ *and* $\Delta = \{v(p^n) \mid n < \omega\}$ *if* $\mathrm{char}(R) = p > 0$*).*

IN equicharacteristic $p > 0$, one does not obtain relative quantifier elimination in the RV-language. Some more hypotheses are required to ensure that the model theory is tractable.

Definition 5.4.2. — Let k be a valued field with valuation v, residue field R and value group V. We say that (k, v) is **Kaplansky** if either $\mathrm{char}(R) = 0$ or $\mathrm{char}(R) = p > 0$ and,

1. R is perfect,
2. R admits no finite separable extension R' such that p divides $[R' : R]$,
3. $pV = V$.

A valued field (k, v) is **algebraically maximal** if it admits no immediate algebraic extension.

ALGEBRAICALLY maximal Kaplansky fields are perfect and henselian, hence large. Moreover, they admit quantifier elimination in the RV-language (something which is not true in general for henselian fields in equicharacteristic $p > 0$). This is [HH18], Corollary A.3.

PROPOSITION 5.4.3. — *Suppose* (k, v) *is algebraically maximal and Kaplansky. Then* k *admits elimination of the field quantifier in the* RV*-language (where* $\Delta = \{0\}$*).*

IN particular,

COROLLARY 5.4.4. — *The following fields are éz:*

1. (k, v) *henselian,* $\operatorname{char}(k) = 0$*,*
2. (k, v) *algebraically maximal and Kaplansky.*

Epilogue: acknowledgements

> Many people could say things in a cutting way, Nanny knew. But Granny Weatherwax could listen in a cutting way. She could make something sound stupid just by hearing it.
>
> Terry Pratchett

NOW that the story has been told, the curtains are quietly closing down and actors have a brief moment of peace to catch their breath, some words of thanks.

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Dolor hic tibi proderit olim.

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Appendix: *algebraic geometry and valuation theory*

Hic sunt dracones: modern algebraic geometry

EACH story requires a proper world-building, a cosmogony of the world your adventures will take place in. This appendix will try to sketch out, briefly, the geography of the vast land of modern algebraic geometry and the linguistics of scheme theory. A warning: these pages don't tell the full story, and aren't trying to do so; I owe my very limited knowledge of algebraic geometry to the wonderful [GW10], and I encourage you to refer to it to clear any of the inevitable obscure points of this exposition.

A.1 Definitions and basic results

SOME preliminary knowledge of sheaves will be assumed. The salient points can be found in Görtz-Wedhorn.

Definition A.1.1. — A **locally ringed space** is the datum of a topological space X together with a sheaf $\mathcal{O}_X$ of rings such that, for each $p \in X$, the stalk $\mathcal{O}_{X,p}$ is a local ring (called the **local ring** at p). Its maximal ideal will be denoted by $\mathfrak{m}_p$ and the quotient $\kappa(p) := \frac{\mathcal{O}_{X,p}}{\mathfrak{m}_p}$ will be called the **residue field** at p.

AS an aside, notice that if you have a section $f \in \mathcal{O}_X(U)$, then f "takes values" on U in the following sense: to each $p \in U$, we can consider $f_p \in \mathcal{O}_{X,p}$ and define

$$f(p) := (f_p \text{ mod. } \mathfrak{m}_p) \in \kappa(p).$$

As you can see, this does not define *functions* in a meaningful way; if anything, it defines multi-valued functions. But nevertheless it is enough to be able to say "$f \in \mathcal{O}_X(U)$ vanishes at $p \in U$" and similar expressions.

ONCE you have locally ringed spaces, you need morphisms; since such a datum combines together topological and algebraic information, morphisms should preserve both.

Definition A.1.2. — A **morphism** of locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is the datum of two morphisms $(f, f^\flat)$, where $f : X \to Y$ is a continuous map and $f^\flat : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is a morphism of sheaves of rings, such that the *adjoint* map to $f^\flat$, $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$, induces a morphism of local rings at the level of stalks, i.e. $f_p^\sharp : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ sends the maximal ideal to the maximal ideal.

THIS definition is a bit of a mouthful, but the gist is that f preserves the topological information, whereas $f^\flat$ – or actually its evil twin, $f^\sharp$ – preserves the local ring structure that lives hidden in the stalks of the sheaf.

IF, like me, you started your journey in algebraic geometry in the comforting port of "classical" algebraic geometry, you already have a locally ringed space available at the back of your mind. Indeed, if you take an affine variety X over an algebraically closed field k and pick a point $p \in X$, you can consider its local ring $\mathcal{O}_{X,p}$ whose maximal ideal is exactly that of regular functions that vanish at p. In this case, $\kappa(p) \cong k$.

THE canonical example of locally ringed space is the spectrum of a ring. If A is a commutative ring, then $\operatorname{Spec}(A)$ is the topological spaces whose underlying set is the set of prime ideals of A and the topology has, as closed sets, precisely the subsets of $\operatorname{Spec}(A)$ of the form

$$V(M) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid M \subseteq \mathfrak{p}\}$$

for some $M \subseteq A$. For any $f \in A$, let $D(f) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid f \notin \mathfrak{p}\}$. We can now assign a sheaf of rings to $X = \operatorname{Spec}(A)$ by declaring that $\mathcal{O}_X(D(f)) = A_f$ for any $f \in A$; one can check that, since the sets of the form $D(f)$ (the **principal open subsets**) form a basis for the open subsets of X, this data is enough to build a sheaf on X. In particular, $\mathcal{O}_{X,p} = A_p$ and hence this sheaf turns X into a locally ringed space. We shall call this the structure sheaf on $\operatorname{Spec}(A)$.

THE equality in the last sentence of the example is the first instance of a possible confusion that might ensue later on; since the points of $\operatorname{Spec}(A)$ are the prime ideals of A, the point p has two distinct roles in the equality: on the left, it is a point of $\operatorname{Spec}(A)$; on the right, it is a prime ideal $p \subseteq A$.

FROM this canonical example we build our first definition of scheme. Notice that, once we have given objects and morphisms, we can consider the category of locally ringed spaces and we thus get a notion of "isomorphism" in that category.

Definition A.1.3. — An **affine scheme** is a locally ringed space isomorphic to the spectrum of a ring together with its structure sheaf. Denote by (**Aff**) the category of affine schemes (where the morphisms are those of locally ringed spaces).

A short, but important, remark: a map between rings $A \to B$ induces a morphism of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ by taking preimages of prime ideals. This identifies a (contravariant) functor from the category (**Comm**) of commutative rings to (**Aff**) which is actually an (anti)equivalence of categories. In particular, if $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, then

$$\operatorname{Hom}_{(\mathbf{Aff})}(X, Y) \cong \operatorname{Hom}_{(\mathrm{Comm})}(B, A).$$

THIS allows us to consider an important geometric object under this new light; if A is a ring, let

$$\mathbb{A}^n_R = \operatorname{Spec} R[t_1, \dots t_n]$$

be the **n-dimensional affine space** over R. If it is clear from the context, the subscript R is dropped and I will write $\mathbb{A}^n$. Many more examples could be introduced, but we shall not need them right now because these couple of definitions already enable us to introduce a very general object.

Definition A.1.4. — A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ that admits an open covering

$$X = \bigcup_{i \in I} X_i$$

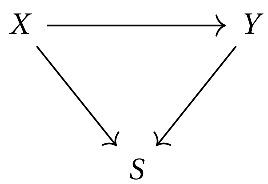
such that, for each $i \in I$, $(X_i, \mathcal{O}_X|_{X_i})$ is an affine scheme. A morphism of schemes is a morphism of locally ringed spaces, so we get a category (**Sch**).

THE remark we made above carries over with some restrictions. Suppose you have schemes $(X, \mathcal{O}_X)$ and $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, then

$$\operatorname{Hom}_{(\mathbf{Sch})}(X, \operatorname{Spec} A) \cong \operatorname{Hom}_{(\mathbf{Comm})}(A, \mathcal{O}_X(X)).$$

FOR most of the time, we will need to work in a "geometric" setting, i.e. in some sense *over a field*. This can be made precise in the following way.

Definition A.1.5. — A **scheme over S**, where S is another scheme, is a morphism of schemes $X \to S$. This can be turned into a category (**Sch**/S) by saying that a morphism of schemes over S is a morphism of schemes $X \to Y$ such that



commutes. If $S = \operatorname{Spec}(R)$, we also say X is a scheme over R, or an R-scheme.

WE are one step closer to varieties in a very concrete, geometric sense. We want to recover some of the properties that "classical" varieties had, so we introduce some terminology.

Definition A.1.6. — If k is a field and $X \to \operatorname{Spec} k$ is a k-scheme, we say that X is **locally of finite type** if there is an affine open cover $X = \bigcup_{i \in I} X_i$ such that, for each $i \in I$, $X_i = \operatorname{Spec} A_i$ with A_i a finitely generated k-algebra. If (the underlying topological space of) X is also compact (or, as algebraic geometers seem to be very fond of saying, “quasicompact”), we say X is **of finite type**.

THE requirement of being “locally of finite type” is akin to working with manifolds that are locally homeomorphic to *finite-dimensional* Euclidean spaces.

A.1.1 *Fiber products and base change*

WE need a final ingredient, whose name sometimes makes people flinch: fiber products. In the classical case, fiber products are built through fiber products of sets: if $f : X \to Y$ and $g : Z \to Y$ are maps, then one can build $X \times_Y Z$ by taking

$$X \times_Y Z = \{(x,z) \in X \times Z \mid f(x) = g(z)\}.$$

One can show that if X, Y, Z are affine varieties and f, g are morphisms, then $X \times_Y Z$ is again a variety, although it might end up not being affine (it is, in general, quasi-projective). The issue presents itself in the general case and is best explored in the following section.

FOR now, we sketch out how the fiber product is built when X, Y, Z are affine schemes.

Definition A.1.7. — Let $X \to Y$, $Z \to Y$ be schemes over Y. The **fiber product** is the datum of a scheme $X \times_Y Z$ together with morphisms $p : X \times_Y Z \to X$, $q : X \times_Y Z \to Z$ that make this diagram commute:

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{p} & X \\ \downarrow{\scriptstyle q} & \lrcorner & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

and such that they satisfy the following universal property: if W is another scheme with morphisms $\psi : W \to X, \varphi : W \to Z$ that make the diagram

$$\begin{array}{ccc} W & \xrightarrow{\psi} & X \\ \downarrow{\scriptstyle \varphi} & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

commute, then there is a morphism $\eta : W \to X \times_Y Z$ such that $p \circ \eta = \psi$ and $q \circ \eta = \varphi$.

DUE to its universal property, the fiber product is precisely the product in the category $(\mathbf{Sch}/Y)$. Its existence is not immediate: suppose we had affine schemes

$$X = \operatorname{Spec} A \to Z = \operatorname{Spec} R,\ Y = \operatorname{Spec} B \to Z = \operatorname{Spec} R,$$

then the affine scheme $W = \operatorname{Spec}(A \otimes_R B)$, together with the morphisms of affine schemes induced by the ring morphisms $A \to A \otimes_R B$, $B \to A \otimes_R B$, satisfies the universal property. One can use this fact to deduce the more general one,

THEOREM A.1.8. — *The fiber product of Y-schemes exists for any scheme Y.*

TAKE the particular case where $X \to Y$ is an Y-scheme and we have a morphism $Y' \to Y$. We can form the fiber product $X \times_Y Y'$, which is then (besides being an Y-scheme) an Y'-scheme called the **base change** of X by the morphism $Y' \to Y$. If $f : X \to Z$ is a morphism of Y-schemes, the morphism $f \times_Y \mathrm{id}_{Y'} : X \times_Y Y' \to Z \times_Y Y'$ is a morphism of Y'-schemes, also called **base change**. This allows to formalize the idea of changing base field for a k-scheme: if $k \to k'$ is a field extension and $X \to \operatorname{Spec} k$ is a k-scheme, then $X_{k'} = X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ is the **base change** of X to k.

FINALLY, we introduce the notion of separated k-scheme. Fix a k-scheme X.

Definition A.1.9. — The **diagonal morphism** of X is the morphism of k-schemes

$$\Delta_{X/k} = (\mathrm{id}_X, \mathrm{id}_X) : X \to X \times_k X.$$

TO motivate what will follow, recall that for a topological space X being Hausdorff is equivalent to the diagonal being closed in X^2. Schemes will not, in general, be Hausdorff, but we can identify those schemes who are close enough.

Definition A.1.10. — A k-scheme X is said to be **separated** if the diagonal morphism $\Delta_{X/k}$ is a closed immersion, i.e. the continuous function maps X homeomorphically to a closed subset of $X \times_k X$ and the sheaf morphism

$$\Delta^\flat_{X/k} : \mathcal{O}_{X \times_k X} \to (\Delta_{X/k})_* \mathcal{O}_X$$

is surjective.

A.1.2 *k-varieties and rational points*

WE have all the necessary ingredients to introduce a more familiar notion (although the identification with "classical" notions will be somehow tricky).

Definition A.1.11. — A k-scheme $X \to \operatorname{Spec} k$ is said to be a **k-variety** if it is separated and of finite type. Denote the category of k-varieties by $(\mathbf{Var}_k)$. A k-variety that is also an affine scheme is an **affine k-variety**. Their category is denoted by $(\mathbf{AffVar}_k)$.

VARIETIES enjoy important closure properties.

PROPOSITION A.1.12. — *The fiber product of k-varieties over a k-variety is a k-variety.*

AT this point, one could wonder what has happened to our intuition of "varieties" as sets of common zeroes of polynomials in affine or projective space. Given a variety in this sense, defined by polynomials with coefficients from a field k, one could magick out a k-variety using, as the sheaf of rings, precisely the rings of regular functions on the variety; the viceversa is generally not true. Two more properties are required: that the scheme be reduced[1] and irreducible. Under these two constraints, the categories of "classical varieties" (sometimes called prevarieties, for example in [GW10]) and k-varieties are equivalent.

BUT despite all this, the questions still stands: what happened to the points? We might want to ask about points with "coordinates" in a larger field, but we don't have an obvious notion of coordinates available. Schemes are big, complicated objects: take for example $\operatorname{Spec}\mathbb{Z}$. It contains several "closed" points, the prime ideals of the form (p) for $p \neq 0$: for all these, $\overline{\{(p)\}} = \{(p)\}$. But the space is not T1: the point (0) is not only not closed, but actually $\overline{\{(0)\}} = \operatorname{Spec}\mathbb{Z}$. We call such a point **generic**, and all the other points are its **specializations**. Even if we restrict to k-varieties – instead of general schemes – things can get messy. Nevertheless, we recover the notion of points with "coordinates" in a certain field.

Definition A.1.13. — Suppose X is a k-variety. If $k \subseteq K$ is a field extension, then $\operatorname{Spec} K$ is a k-scheme and we can consider the set of morphisms (in the category of k-schemes) from $\operatorname{Spec} K$ to X. Denote it by

$$X(K) = \operatorname{Hom}_k(\operatorname{Spec} K, X)$$

and call it the set of **K-valued points** of X.

IN the case where k is algebraically closed, k-valued points $x : \operatorname{Spec} k \to X$ identify closed points of X and establish a bijection between $X(k)$ and the closed points of X. If X is reduced and irreducible, this is a bijection between $X(k)$ and the associated "classical" variety.

THIS definition may seem weird at first, so a sanity check is due: take affine n-space $\mathbb{A}^n_k$, then

$$\mathbb{A}^n_k(k) = \operatorname{Hom}_k(\operatorname{Spec} k, \mathbb{A}^n_k) \cong \operatorname{Hom}_{(\mathrm{Comm})}(k[t_1, \dots t_n], k) \cong k^n,$$

where the latter bijection is given by $\varphi \mapsto (\varphi(t_1), \dots \varphi(t_n))$.

[1] Reduced will be defined later on; if you are familiar with classical algebraic varieties, the point is that we want to avoid nilpotents in the sections of the sheaf.

NOW, consider a k-variety $X \to \operatorname{Spec} k$. For any of its points $x \in X$, consider an affine open neighbourhood $U = \operatorname{Spec} A$ so that $x \in U$ corresponds to the prime ideal $\mathfrak{p} \subseteq A$. We have a natural morphism $A \to A_\mathfrak{p} = \mathcal{O}_{U,x} = \mathcal{O}_{X,x}$, that gives rise to a morphism of schemes $\operatorname{Spec} \mathcal{O}_{X,x} \to \operatorname{Spec} A = U \subseteq X$. The quotient map $\mathcal{O}_{X,x} \to \kappa(x)$ induces a morphism of schemes $\operatorname{Spec} \kappa(x) \to \operatorname{Spec} \mathcal{O}_{X,x}$, so we get a chain of morphisms

$$\operatorname{Spec} \kappa(x) \to \operatorname{Spec} \mathcal{O}_{X,x} \to X \to \operatorname{Spec} k$$

that, at the ring level, induces a field extension $k \to \kappa(x)$.

THIS field extension encodes a lot of information about the point x.

PROPOSITION A.1.14. — *The point x is closed if and only if $k \to \kappa(x)$ is a finite extension.*

MOREOVER, it allows us to say that a point is **rational** if the field extension $k \to \kappa(x)$ is an isomorphism.

IF k is algebraically closed, all closed points are k-rational, so we get back the idea that closed points of k-varieties that are reduced and irreducible over an algebraically closed field are precisely the good ol' tuples of elements of the field satisfying certain polynomial relations.

THIS notion somehow clashes with the notion of k-valued points defined above, in that they both seem to concretize the idea of point with "coordinates" in k. However, this is not a real clash – the assignment $(x : \operatorname{Spec} k \to X) \mapsto x$ gives a bijection between $X(k)$ and the k-rational points of X, so we will more often than not identify the two notions.

A.1.3 *Subschemes and immersions*

THE notion of "open" subscheme will be rather straightforward; as expected, the structure sheaf behaves nicely when we try to restrict it to open sets.

Definition A.1.15. — Suppose $(X, \mathcal{O}_X)$ is a scheme and $U \subseteq X$ is an open subset. Then the locally ringed space $(U, \mathcal{O}_X|_U)$ is called an **open subscheme** of X.

NOW, technically we still need to show that $(U, \mathcal{O}_X|_U)$ is not *any* locally ringed space, but a scheme.

LEMMA A.1.16. — *$(U, \mathcal{O}_X|_U)$ is a scheme. If X is affine, so is U.*

Proof. By definition of scheme, $X = \bigcup_{i\in I} X_i$ where each X_i is an affine scheme. Moreover, each X_i has a basis of its topology given by principal open sets that are themselves affine schemes[2]; in particular, U itself is covered by affine schemes and is thus a scheme. ■

[2] The image you should have in mind is that $\mathbb{A}^1_k \setminus \{(0)\}$ is isomorphic to the hyperbole in the plane.

NOTICE that the inclusion map $i : U \hookrightarrow X$ induces a morphism of sheaves in the following way: for any open $V \subseteq X$, we have a map

$$\mathcal{O}_X(V) \to \mathcal{O}_X(V \cap U) = \mathcal{O}_X|_U(i^{-1}(V)) = j_*\mathcal{O}_X|_U(V)$$

and so we get a morphism $i^\flat : \mathcal{O}_X \to i_*\mathcal{O}_X|_U$ of sheaves. In particular, $(i, i^\flat)$ is a morphism of schemes $U \to X$. This clears the way for a more general notion,

Definition A.1.17. — A morphism $i : X \to Y$ of schemes is an **open immersion** if the continuous map is an homeomorphism $X \xrightarrow{\sim} i(X) \subseteq Y$ where $i(X)$ is open in Y and the sheaf morphism $i^\flat : \mathcal{O}_X \to i_*\mathcal{O}_Y$ induces a sheaf isomorphism $\mathcal{O}_Y|_{i(X)} \cong i_*\mathcal{O}_X$.

ON the other hand, closed subschemes are slightly harder to work with. If we restrict ourselves to an affine scheme $X = \operatorname{Spec} A$, then the "ideal" (pun not intended) closed subscheme is something of the form $\operatorname{Spec} A/\mathfrak{a}$, where $\mathfrak{a}$ is some ideal in A, since it is homeomorphic to the closed subspace $V(\mathfrak{a}) \subseteq \operatorname{Spec} A$.

Definition A.1.18. — If $(X, \mathcal{O}_X)$ is a scheme, a subsheaf $\mathcal{I} \subseteq \mathcal{O}_X$ is called a **sheaf of ideals** if $\mathcal{I}(U)$ is an ideal in $\mathcal{O}_X(U)$ for every open $U \subseteq X$. In this case, the presheaf $U \mapsto \mathcal{O}_X(U)/\mathcal{I}(U)$ can be sheafified to a sheaf called the **quotient sheaf**.

SINCE we want closed subschemes to be essentially of the form $\operatorname{Spec} A/\mathfrak{a}$, a good starting point would be that their sheaves be isomorphic to quotient sheaves of the form we have just defined.

Definition A.1.19. — If $(X, \mathcal{O}_X)$ is a scheme, a **closed subscheme** is given by a closed subset $j : Z \hookrightarrow X$ and a sheaf $\mathcal{O}_Z$ on Z such that

1. $(Z, \mathcal{O}_Z)$ is a scheme,

2. $j_*\mathcal{O}_Z$ is isomorphic to a quotient sheaf $\mathcal{O}_X/\mathcal{I}$ for some sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$.

THIS definition, while slightly miraculous, once again clears the way for a generalization. Notice that there is a canonical surjective projection $\mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}$.

Definition A.1.20. — A morphism $j : Z \to X$ of schemes is called a **closed immersion** if the continuous map is a homeomorphism $Z \xrightarrow{\sim} j(Z) \subseteq X$, where $j(Z)$ is closed in X, and the sheaf morphism $j^\flat : \mathcal{O}_X \to j_*\mathcal{O}_Z$ is surjective.

THIS recovers our intuition, in that sets of the form $\operatorname{Spec} A/\mathfrak{a}$ should be exactly the closed subschemes of $\operatorname{Spec} A$.

THEOREM A.1.21. — *Let* $X = \operatorname{Spec} A$ *be affine. There is a bijective correspondence between the set of ideals of* A *and the closed subschemes of* X*, given by* $\mathfrak{a} \mapsto V(\mathfrak{a})$.

FOR the sake of completeness, there is a more general notion of subscheme and immersion.

Definition A.1.22. — Let $(X, \mathcal{O}_X)$ be a scheme. A **subscheme** of X is a scheme $(Y, \mathcal{O}_Y)$ such that

1. Y is locally closed in X,
2. if $U = X \setminus (\overline{Y} \setminus Y)$, then Y is a closed subscheme of U.

The sinister open subscheme U is precisely the biggest open subset of X in which Y is closed. An **immersion** $i : X \to Y$ is a morphism of schemes such that

1. the continuous map is an homeomorphism between X and $i(X) \subseteq X$ which is locally closed,
2. for all $x \in X$, the morphism of local rings $i_x^\sharp : \mathcal{O}_{Y,i(x)} \to \mathcal{O}_{X,x}$ is surjective.

IF we restrict to k-schemes of finite type, we get that all subschemes are again of finite type over k; moreover,

THEOREM A.1.23. — *Being an open or closed immersion is local on the target (i.e., it can be checked on the preimages of an open cover of the target) and closed under composition.*

BY the way, this allows to reintroduce familiar objects from classical algebraic geometry.

Definition A.1.24. — The **projective space** of dimension n over a ring R is obtained by gluing $n+1$ copies of affine space $\mathbb{A}_R^n$. We denote it by $\mathbb{P}_R^n$.

PROJECTIVE space has some interesting properties (for example, $\mathcal{O}_{\mathbb{P}_R^n}(\mathbb{P}_R^n) \cong R$), but more importantly it allows us to recover the classical notions of projective and quasi-projective variety.

Definition A.1.25. — A **projective** k-variety is a k-variety together with a closed immersion into $\mathbb{P}_k^n$. A **quasi-projective** k-variety is a k-variety together with an immersion into $\mathbb{P}_k^n$.

A.2 Smoothness

A.2.1 *Commutative algebra interlude: differentials*

LET $\varphi : R \to S$ be a ring map and suppose M is a S-module. What follows mostly comes from either [Jon] or [Gro64b].

Definition A.2.1. — An R-**derivation** into M is a map $D : S \to M$ that is additive, zero on $\varphi(R)$ and satisfies $D(ab) = aD(b) + bD(a)$.

CONSIDER the S-module $\mathrm{Der}_R(S, M)$ of all R-derivations on M. The association $M \mapsto \mathrm{Der}_R(S, M)$ is actually functorial, acting on morphisms by

$$(\alpha : M \to N) \mapsto (\alpha \circ D : R \to N).$$

THERE is a map of free S-modules

$$\bigoplus_{(a,b)\in S^2} S[(a,b)] \oplus \bigoplus_{(f,g)\in S^2} S[(f,g)] \oplus \bigoplus_{r\in R} S[r] \to \bigoplus_{a\in S} S[a]$$

defined by

$$[(a,b)] \mapsto [a+b] - [a] - [b], [(f,g)] \mapsto [fg] - f[g] - g[f], [r] \mapsto [\varphi(r)]$$

and we denote by $\Omega_{S/R}$ its cokernel. The quotient map $d : S \to \Omega_{S/R}$ will map $a \mapsto da = \overline{[a]}$. In particular, d is a derivation.

Definition A.2.2. — The pair $(\Omega_{S/R}, d)$ is called the **module of differentials**.

THIS is usually called the module "of Kähler differentials". Unless confusion ensues, I will refrain from using this name, due to Erich Kähler's problematic history as a Nazi supporter during WWII and as a Third Reich apologist afterwards.

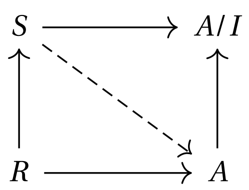
THE module of differentials satisfies an universal property, namely that

$$\mathrm{Hom}_S(\Omega_{S/R}, M) \to \mathrm{Der}_R(S, M), \alpha \mapsto \alpha \circ d$$

is an isomorphism of functors.

SUPPOSE now that $R \to S$ is a ring map, so that S is an R-algebra.

Definition A.2.3. — We say that S is **formally smooth** over R if for every commutative R-algebra A and ideal $I \subseteq A$ such that $I^2 = 0$ and for every R-algebra morphism $S \to A/I$, there is a map $S \to A$ lifting this morphism. In other words, the dotted arrow that makes the diagram



commute exists.

THEOREM A.2.4 *([Gro64a], 19.1.12).* — *Suppose R is a ring, M is a finitely generated R-module, N a projective R-module, $f : M \to N$ a morphism. For* *all prime ideals $\mathfrak{p} \subseteq R$, $f \otimes \mathrm{id} : M \otimes_R \kappa(\mathfrak{p}) \to N \otimes_R \kappa(\mathfrak{p})$ is injective if and only if there are finitely many $x_1, \dots x_m \in M$ such that $M_{\mathfrak{p}} = \langle \overline{x_1}, \dots \overline{x_m} \rangle$ and finitely many linear forms $y_1, \dots y_m$ on N such that*

$$\det(y_i(f(x_j))_{i,j} \notin \mathfrak{p}.$$

THEOREM A.2.5 *([Gro64a], 22.6.6). — Suppose R is a ring, S is a formally smooth R-algebra, $J \subseteq S$ is an ideal, $T = S/J$. Moreover, suppose J/J^2 is a finitely generated T-module. Then T is a formally smooth R-algebra if and only if, for all primes $\mathfrak{p} \subseteq T$, $T_{\mathfrak{p}}$ is a formally smooth R-algebra.*

THEOREM A.2.6 *([Gro64a], 22.6.4). — Suppose R is a ring, S is a formally smooth R-algebra, $J \subseteq S$ is an ideal, $T = S/J$. Moreover, suppose J/J^2 is a finitely generated T-module. If $\mathfrak{p} \subseteq T$ is a prime ideal and $\kappa(\mathfrak{p})$ is the residue field at $T_{\mathfrak{p}}$, $\mathfrak{p}' \subseteq S$ is the prime corresponding to $\mathfrak{p}$, $\mathfrak{q} \subseteq R$ the prime corresponding to $\mathfrak{p}'$, then $T_{\mathfrak{p}}$ is a formally smooth R-algebra if and only if $T_{\mathfrak{p}}$ is a formally smooth $R_{\mathfrak{q}}$-algebra if and only if the morphism*

$$d|_J \otimes_S \mathrm{id} : (J/J^2) \otimes_C \kappa(\mathfrak{p}) \to \Omega_{S/R} \otimes_S \kappa(\mathfrak{p})$$

is injective.

COROLLARY A.2.7. — *Suppose R is a ring, S a formally smooth R-algebra through a morphism f, $T = S/J$ for some ideal $J \subseteq S$ such that J/J^2 is a finitely generated T-module. Then T is a formally smooth R-algebra if and only if, for all primes $\mathfrak{p} \subseteq T$, there are finitely many $x_1, \dots x_m \in J$ such that $(J/J^2)_{\mathfrak{p}} = \langle \overline{x_1}, \dots \overline{x_m} \rangle$ and finitely many linear forms $y_1, \dots y_m$ on $\Omega_{S/R}$ such that*

$$\det(y_i(f(x_j)))_{i,j} \notin \mathfrak{p}.$$

A.2.2 *Smooth morphisms of k-varieties*

Definition A.2.8. — A point $p \in X$ is said to be **nonsingular** (or **regular**) if $\dim_{\kappa(p)} \frac{\mathfrak{m}_p}{\mathfrak{m}_p^2} = \dim \mathcal{O}_{X,p}$. The set of regular points of X is called X_{reg}.

Definition A.2.9. — A morphism $f : X \to Y$ is said to be **smooth of relative dimension** d at $p \in X$, where $d \geq 0$ is an integer, if there exist affine open neighbourhoods $U \subseteq X$ of p and $V = \operatorname{Spec} R \subseteq Y$ of $f(p)$ and an open subscheme $W \subseteq \operatorname{Spec} R[x_1, \dots x_{d+r}]/(f_1, \dots f_r)$ such that the following diagram commutes,

$$\begin{array}{ccccc} U & \xrightarrow{\cong} & W & \hookrightarrow & \operatorname{Spec} R[x_1, \dots x_{d+r}]/(f_1, \dots f_r) \\ \downarrow{\scriptstyle f|_U} & & \downarrow{\scriptstyle \rho|_W} & \swarrow{\scriptstyle \rho} & \\ V & \xrightarrow{\cong} & \operatorname{Spec} R & & \end{array}$$

and $\det\left(\frac{\partial f_j}{\partial x_i}\right)_{i,j\le r}$ is nowhere zero on W.

Definition A.2.10. — A point $p \in X$ is said to be **smooth** (of relative dimension d) if the structure morphism $X \to \operatorname{Spec} k$ is smooth (of relative dimension d) at p. Denote the set of smooth points by X_{sm}.

THEOREM A.2.11. — *Suppose p is a k-rational point, i.e. $\kappa(p) = k$. Then p is regular and $\mathcal{O}_{X,p}$ has dimension d if and only if p is smooth (of relative dimension d). In particular,*

$$X_{\mathrm{reg}}(k) = X_{\mathrm{sm}}(k).$$

THEOREM A.2.12. — *A morphism $\varphi : X \to Y$ is smooth at $p \in X$ if and only if there are an affine neighbourhood $U = \operatorname{Spec} S$ around p and an affine neighbourhood $V = \operatorname{Spec} R$ around $\varphi(p)$ such that the induced ring map $R \to S$ is formally smooth and of finite presentation, i.e. there exist integers $n, m \in \mathbb{N}$ and polynomials $f_1, \dots f_m \in R[x_1, \dots x_n]$ such that $S \cong R[x_1, \dots x_n]/(f_1, \dots f_m)$ as R-algebras.*

A sketch of valuation theory

THE material covered in this chapter comes from [EP05].

B.1 Valuations, valuation rings and places

Definition B.1.1. — Let k be a field and Γ be an ordered abelian group. A **valuation** v on k is a surjective map $v : k \to \Gamma \cup \{\infty\}$ such that, for all $x, y \in \Gamma$,

1. $v(x) = \infty$ implies $x = 0$,
2. $v(xy) = v(x) + v(y)$,
3. $v(x + y) \geq \min(v(x), v(y))$.

Its **rank** is the rank of the **value group** Γ, i.e. the order-type of the collection of proper convex subgroups of Γ.

IF $\Gamma = \{0\}$, we obtain the **trivial valuation**; if Γ has rank 1, then it can be observed that $\Gamma \leq (\mathbb{R}, +)$.

OUT of a **valued field** $(k, v) = (k, v, \Gamma)$ we can build $\mathcal{O}_v = \{x \in k \mid v(x) \geq 0\}$, the **valuation ring** of (k, v). It is a local ring with maximal ideal $\mathcal{M}_v$, and the residue field $\overline{k}_v = \mathcal{O}_v / \mathcal{M}_v$ is called the **residue field** of (k, v). Sometimes, the value group and residue field of (k, v) are denoted, respectively, by vk and kv.

VALUATION rings often appear in other settings (for example, in the theory of DVRs, discrete valuation rings). A **valuation ring** of k is often defined as a subring $\mathcal{O}$ of k such that, for all $x \in k^\times$, either $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$. Then,

PROPOSITION B.1.2. — *Suppose $\mathcal{O} \subseteq k$ is a valuation ring, then there exists a valuation v on k such that $\mathcal{O} = \mathcal{O}_v$.*

THE value group is precisely the quotient $\Gamma := k^\times / \mathcal{O}^\times$, which can be rewritten additively and ordered by $x\mathcal{O}^\times \leq y\mathcal{O}^\times$ if and only if $\frac{y}{x} \in \mathcal{O}$. At this point, the valuation can be defined by $v(x) = x\mathcal{O}^\times \in \Gamma$. Moreover, the unique maximal ideal of $\mathcal{O}$ is exactly $\mathcal{M} := \mathcal{O} \setminus \mathcal{O}^\times$. The trivial valuation is then determined by $\mathcal{O} = k$.

Definition B.1.3. — Two valuations $v_i : k \to \Gamma_i \cup \{\infty\}$, for $i \in \{1, 2\}$, are said to be **equivalent** if $\mathcal{O}_{v_1} = \mathcal{O}_{v_2}$. Equivalently, if there is an order-preserving isomorphism $\rho : \Gamma_1 \to \Gamma_2$ such that $\rho \circ v_1 = v_2$.

This establishes a bijection

$$\{\text{valuation rings of } k\} \longleftrightarrow \frac{\{\text{valuations on } k\}}{\sim}$$

where $v_1 \sim v_2$ if they are equivalent (i.e. up to order-preserving isomorphism of the value groups).

THERE is one more gadget that turns out to be essentially equivalent to a valuation.

Definition B.1.4. — Suppose k and K are fields. A map $\varphi : k \to K \cup \{\infty\}$ is a **place** of k if, for all $x, y \in k$, $\varphi(x+y) = \varphi(x)+\varphi(y)$, $\varphi(x\cdot y) = \varphi(x)\cdot\varphi(y)$, $\varphi(1) = 1$.

FOR any place φ on k, $\mathcal{O} = \varphi^{-1}(K)$ is a valuation ring of k whose maximal ideal is $\mathcal{M} = \varphi^{-1}(\{0\})$ and whose residue field is $\varphi(\mathcal{O})$. Viceversa, for every valuation ring $\mathcal{O}$ of k with maximal ideal $\mathcal{M}$, the map

$$\varphi(x) = \begin{cases} x+\mathcal{M}, & x \in \mathcal{O}, \\ \infty, & x \in k \setminus \mathcal{O} \end{cases}$$

defines a place $\varphi : k \to \frac{\mathcal{O}}{\mathcal{M}} \cup \{\infty\}$.

Definition B.1.5. — If $k \subseteq F$, then a place φ on F is a **k-place** if it is the identity on k.

Definition B.1.6. — Two places on the same field are **equivalent** if there is an isomorphism of their residue fields that commutes with the places.

B.1.1 *Extending valuations*

SUPPOSE you have a field extension $k_1 \subseteq k_2$ and two valued fields $(k_1, \mathcal{O}_1)$ and $(k_2, \mathcal{O}_2)$. We say that $\mathcal{O}_2$ is a **prolongation**, or an **extension**, of $\mathcal{O}_1$ if $\mathcal{O}_2 \cap k_1 = \mathcal{O}_1$. We write $(k_1, \mathcal{O}_1) \subseteq (k_2, \mathcal{O}_2)$. The classical case is that of a field extension $k_1 \subseteq k_2$ and a valuation ring on k_2, $\mathcal{O}_2$; in this scenario, $\mathcal{O}_1 := k_1 \cap \mathcal{O}_2$ is again a valuation ring and $\mathcal{O}_2$ extends $\mathcal{O}_1$.

THE other way around is trickier, and uses the following theorem, also known as *Chevalley's extension theorem.*

THEOREM B.1.7. — *Suppose k is a field, $R \subseteq k$ is a subring and $\mathfrak{p} \subseteq R$ is a prime ideal. Then there exists a valuation ring $\mathcal{O}$ of k such that $R \subseteq \mathcal{O}$ and $\mathcal{M} \cap R = \mathfrak{p}$, where $\mathcal{M} \subseteq \mathcal{O}$ is the maximal ideal.*

COROLLARY B.1.8. — *Suppose $k_1 \subseteq k_2$ is a field extension and $\mathcal{O}_1 \subseteq k_1$ is a valuation ring. Then there is an extension $\mathcal{O}_2$ of $\mathcal{O}_1$ in k_2.*

THE consequences of Chevalley's theorem are more far-reaching. For example, COROLLARY B.1.9. — *Every valuation ring $\mathcal{O} \subseteq k$ is integrally closed in k.*

COROLLARY B.1.10. — *If $k \subseteq K$ is a field extension and $\mathcal{O}'$ is a valuation ring of K, then every valuation ring $\mathcal{O} \supseteq \mathcal{O}' \cap k$ can be extended to a valuation ring $\mathcal{O}'' \supseteq \mathcal{O}'$ of K.*

B.1.2 *The algebraic case*

IF $(k_1, \mathcal{O}_1) \subseteq (k_2, \mathcal{O}_2)$, then to each $i = 1, 2$ is associated a valuation $v_i : k_i \to \Gamma_i \cup \{\infty\}$ with $\Gamma_i \cong k_i^\times / \mathcal{O}_i^\times$. Since $\mathcal{O}_1^\times = \mathcal{O}_2^\times \cap k_1^\times$, the composition of $k_1^\times \to k_2^\times$ with the projection $k_2^\times \to k_2^\times / \mathcal{O}_2^\times \cong \Gamma_2$ descends to a morphism

$$\Gamma_1 \cong k_1^\times / \mathcal{O}_1^\times \to k_2^\times / \mathcal{O}_1^\times \cong \Gamma_2,$$

hence we can assume $\Gamma_1 \leq \Gamma_2$.

Definition B.1.11. — Let $e := [\Gamma_2 : \Gamma_1]$. We call it the **ramification index** of this extension.

SIMILARLY, we can regard the residue field $\overline{k_1}$ as a subfield of $\overline{k_2}$.

Definition B.1.12. — Let $f := [\overline{k_2} : \overline{k_1}]$. We call it the **residue degree** of the extension.

Definition B.1.13. — An extension such that $e = f = 1$ is called **immediate**.

THEOREM B.1.14. — *Suppose $(k_1, \mathcal{O}_1) \subseteq (k_2, \mathcal{O}_2)$ is such that $k_1 \subseteq k_2$ is algebraic. Then Γ_2/Γ_1 is a torsion group and $\overline{k_1} \subseteq \overline{k_2}$ is an algebraic extension. Moreover, Γ_1 and Γ_2 have the same rank.*

FINALLY, if we consider an algebraic extension $k_1 \subseteq k_2$ and let

$$k_2 \cap k_1^s = \{x \in k_2 \mid x \text{ is separable over } k_1\}$$

and denote by $[k_2 : k_1]_s = [k_2 \cap k_1^s : k_1]$, then

THEOREM B.1.15. — *Suppose $k_1 \subseteq k_2$ is algebraic and $[k_2 : k_1]_s < \infty$. If $\mathcal{O}$ is a valuation ring of k_1, then*

$$\#\{\textit{prolongations of } \mathcal{O} \textit{ to } k_2\} \leq [k_2 : k_1]_s.$$

NOTICE that

$$[k_2 : k_1] = [k_2 : k_2 \cap k_1^s] \cdot [k_2 : k_1]_s$$

and we denote by $[k_2 : k_1]_i = [k_2 : k_2 \cap k_1^s]$. We call the extension **purely inseparable** if $[k_2 : k_1]_i = 1$.

COROLLARY B.1.16. — *If $k_1 \subseteq k_2$ is purely inseparable, then every valuation ring $\mathcal{O}$ of k_1 extends uniquely to k_1.*

COROLLARY B.1.17. — *Suppose k is separably closed and $\mathcal{O} \subsetneq k$ is a valuation ring. Then there is a unique $\overline{\mathcal{O}} \subseteq k^{\mathrm{alg}}$ and the extension is immediate. In particular, the residue field of $\mathcal{O}$ is algebraically closed and the value group Γ is divisible.*

IF, moreover, we assume that $k \subseteq K$ is finite, then let $n = [K : k]$ and denote by r the number of prolongations of a fixed valuation ring $\mathcal{O} \subseteq k$ to K.

THEOREM B.1.18. — *Let $\mathcal{O}_1, \ldots \mathcal{O}_r$ be said prolungations of $\mathcal{O}$ to K. Then,*

$$\sum_{i=1}^{r} e(\mathcal{O}_i/\mathcal{O}) f(\mathcal{O}_i/\mathcal{O}) \leq n.$$

B.2 Henselian fields

Definition B.2.1. — A valued field $(k, \mathcal{O})$ is **henselian** if $\mathcal{O}$ has a unique prolongation to every algebraic extension $k \subseteq K$.

AS an example, consider a rank-one valuation v on a complete field k.

LEMMA B.2.2. — *Suppose $(k_1, \mathcal{O}_1) \subseteq (k_2, \mathcal{O}_2)$ is an algebraic valued field extension. If $(k_1, \mathcal{O}_1)$ is henselian, then so is $(k_2, \mathcal{O}_2)$.*

LEMMA B.2.3. — *A valuation ring $\mathcal{O}$ is henselian if and only if it extends uniquely to k^s.*

THEOREM B.2.4. — *Suppose $(k, \mathcal{O})$ is a valued field with maximal ideal $\mathcal{M}$, residue field $\overline{k}$, and let $v : k \to \Gamma \cup \{\infty\}$ be the associated valuation. Denote by $f \mapsto \overline{f}$ the map $\mathcal{O}[X] \to \overline{k}[X]$ induced by the residue map $\mathcal{O} \to \overline{k}$. Then, equivalently,*

1. *$(k, \mathcal{O})$ is henselian,*
2. *for each $f \in \mathcal{O}[X]$ and $a \in \mathcal{O}$ such that $\overline{f}(\overline{a}) = 0$ and $\overline{f}'(\overline{a}) \neq 0$, there exists $b \in \mathcal{O}$ such that $\overline{b} = \overline{a}$ and $f(b) = 0$,*
3. *for each $f \in \mathcal{O}[X]$ and $a \in \mathcal{O}$ with $v(f(a)) > 2v(f'(a))$, there exists $b \in \mathcal{O}$ with $f(b) = 0$ and $v(a - b) > v(f'(a))$,*
4. *every polynomial of the form $X^n + a_1 X^{n-1} + \cdots + a_{n-1} X + a_n \in \mathcal{O}[X]$ such that $a_1 \notin \mathcal{M}$ but $a_2, a_3, \ldots a_n \in \mathcal{M}$ has a zero in k.*

COROLLARY B.2.5. — *Suppose $\mathcal{O} \subseteq \mathcal{O}'$ are two valuation rings on k with corresponding maximal ideals $\mathcal{M}' \subseteq \mathcal{M}$. Let $\overline{\mathcal{O}} = \mathcal{O}/\mathcal{M}' \subseteq \overline{k} = \mathcal{O}'/\mathcal{M}'$. Then $(k, \mathcal{O})$ is henselian if and only if both $(k, \mathcal{O}')$ and $(\overline{k}, \overline{\mathcal{O}})$ are henselian.*

B.2.1 *The valuation topology*

GIVEN a valuation $v : k \to \Gamma \cup \{\infty\}$, for each $\gamma \in \Gamma$ and $a \in k$ we define the **open ball** of radius γ around a to be

$$\mathcal{B}_{\gamma}(\gamma) = \{x \in k \mid v(x-a) > \gamma\}.$$

THEOREM B.2.6. — *The subsets of k of the form $\mathcal{B}_{\gamma}(\gamma)$ form a basis of neighbourhoods of a and they thus induce an Hausdorff topology on k.*

WE call this topology the induced topology, or the **valuation topology**.

PROPOSITION B.2.7. — *Sets of the form*

1. $\mathcal{B}_{\gamma}(\gamma)$,
2. $\{x \in k \mid v(x-a) \geq \gamma\}$,
3. $\{x \in k \mid v(x-a) \leq \gamma\}$,
4. $\{x \in k \mid v(x-a) = \gamma\}$,

as γ varies in Γ and $a \in k$, are all clopen subsets in the valuation topology. In particular, $\mathcal{O}$ and $\mathcal{M}$ are clopen.

PROPOSITION B.2.8. — *The field operations are continuous with respect to the valuation topology.*

SAY that two valuation rings $\mathcal{O}_1$ and $\mathcal{O}_2$ on k are **dependent** if $\mathcal{O}_1\mathcal{O}_2$, the smallest subring of k containing both $\mathcal{O}_1$ and $\mathcal{O}_2$, is a proper subring of k. Moreover, if $\mathcal{O} \subseteq k$ is a valuation ring, then any overring $\mathcal{O}' \supseteq \mathcal{O}$ is a valuation ring, called a **coarsening** of $\mathcal{O}$. Then, two dependent valuation rings $\mathcal{O}_1$ and $\mathcal{O}_2$ have a common coarsening (precisely $\mathcal{O}_1\mathcal{O}_2$). Other examples of coarsening come from the localizations $\mathcal{O}_{\mathfrak{p}}$ for a prime ideal $\mathfrak{p} \subseteq \mathcal{O}$, and these are precisely in one-to-one correspondence with the convex subgroups of Γ.

THEOREM B.2.9. — *Two non-trivial valuation rings $\mathcal{O}_1$ and $\mathcal{O}_2$ on k are dependent if and only if they induce the same topology on k.*

IN particular, if $\mathcal{O} \subseteq \mathcal{O}'$ is a non-trivial coarsening, then $\mathcal{O}$ and $\mathcal{O}'$ are dependent and thus they induce the same topology.

B.2.2 *V-topologies*

VALUATION topologies fit into the more general framework of V-topologies.

Definition B.2.10. — A topology τ on k is a **V-topology** if it is a non-discrete field topology and moreover for any neighbourhood U of zero, $(k \setminus U)^{-1}$ is bounded, i.e. for every open neighbourhood V of zero, there is $a \in k^\times$ such that $a \cdot (k \setminus U)^{-1} \subseteq V$.

ORDER, valuation and absolute value topologies are V-topologies. These are, surprisingly, the only ones.

PROPOSITION B.2.11. — *Suppose that τ is a field topology on k. If τ is a V-topology, then it is induced by some absolute value or valuation on k.*

THERE is an analogue of henselianity in this setting, t-henselianity, although one must be aware that the topology induced by a valuation does not contain enough information to reconstruct the valuation itself. In particular, it is not true that a valuation that induces a t-henselian valuation topology is henselian; if anything, because any coarsening of the valuation would induce the same topology.

Definition B.2.12. — A V-topology is **t-henselian** if for any n there is an open neighbourhood U of 0 such that, if $a_0, \dots a_n \in U$, then

$$x^{n+2} + x^{n+1} + a_n x^n + \cdots + a_1 x + a_0$$

has a root in k.

HENSELIAN valuations induce t-henselian topologies. In [ZP78], the authors introduce the notion of t-henselianity and the machinery necessary to deal with topological fields from the model-theoretic perspective. In particular, they prove that a non-separably closed field is t-henselian if and only if it is elementary equivalent – in their language – to an henselian field. More importantly, they show that non-separably closed fields admit at most one t-henselian topology, which we shall call *the* t-henselian topology.

MOREOVER, t-henselian fields satisfy the implicit function theorem.

THEOREM B.2.13. — *Suppose k is t-henselian. Let $f \in k[x_0, \dots x_n, y]$ be a polynomial and suppose $a_0, \dots a_n, b \in k$ are such that $f(a_0, \dots a_n, b) = 0$ and $f_y(a_0, \dots a_n, b) \neq 0$. Then there are two open subsets U, V such that for all $a'_i \in a_i + U$, $i = 0, \dots n$, there is a unique $b' \in b + V$ such that $f(a'_0, \dots a'_n, b') = 0$. Moreover, the map $(a'_0, \dots a'_n) \mapsto b'$ is continuous.*