Canonical bases via jet spaces

in ALFA

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part one:

overview of tools

* canonical bases in ACFA
* jet spaces
* σ-modules

part two:

Zilber's trichotomy

* almost internality
* theorem 1.2 from Pillay-Ziegler
* the trichotomy

NOTATION:

* (K,σ) difference field, $\sigma : K \xrightarrow{\cong} K$
  $\cap|$
  (L,σ), $a \in L$, then $K\langle a\rangle := K(\sigma^n(a) : n \in \mathbb{Z})$

* $\mathrm{Fix}(K) = \{x \in K : \sigma(x) = x\}$, also sometimes $\mathrm{Fix}(\sigma)$

* $a \in K^n$ $\rightsquigarrow$ $\mathrm{Loc}(a/K')$ = smallest K'-variety that contains a
  $K \subseteq K'$
  = "the K'-variety with a as generic point"

* $\overline{Cb(a/K')}$ = smallest acl-closed subset of K' such that $a \underset{Cb(a/K')}{\smile\!\!\!\!\!\!\;|} K'$
  = "weak canonical base"

* K^{alg} = field-theoretic algebraic closure, $acl(X) = acl^{eq}(X)$

Our goal

Y finite-dimensional definable set, with parameters from some algebraically closed difference subfield $k \subseteq \mathcal{U}$, $a \in Y$, $k \subseteq k_1$ another alg. closed difference subfield,

$$c := Cb\left(tp\left(a/k_1\right)\right),$$

then $tp(c/_{k\langle a\rangle})$ is almost internal to the fixed field $Fix(\mathcal{U})$.

this is the "canonical base property"!

Some ideas:

think of zero set of σ-polynomial over k

Y finite-dimensional definable set, with parameters from some algebraically closed difference subfield $k \subseteq \mathcal{U}$, $a \in Y$, $k \subseteq k_1$ another alg. closed difference subfield,

a zero of the σ-polynomial

$$c := \mathrm{Cb}\left(\mathrm{tp}\left({}^{a}/_{k_1}\right)\right),$$

think of the weak canonical basis: generators of the field of definition of $\mathrm{Loc}\left({}^{a}/_{k_1}\right)$

then $\mathrm{tp}\left({}^{c}/_{k\langle a\rangle}\right)$ is almost internal to the fixed field $\mathrm{Fix}(\mathcal{U})$.

up to changing parameters a bit, the information is already seen by σ-polynomials over $\mathrm{Fix}(\mathcal{U})$

PART 1

overview of tools

Canonical bases in $ACFA_0$

$ACFA_0$ - model completion of (inversive) difference fields of characteristic zero (not a complete theory!)

Independence - $(K, \sigma) \models ACFA_0$, $A, B, C \subseteq K$,

$$A \underset{C}{\downarrow} B \iff acl_\sigma(CA) \underset{acl_\sigma(C)}{\overset{ld}{\downarrow}} \underbrace{acl_\sigma(CB)}_{} .$$

$acl_\sigma(CB) = \langle CB \rangle_\sigma^{alg}$

Weak canonical bases - diff. subfield $K \subseteq \mathcal{U} \models ACFA$, $p = tp(a/K)$, then

$\overline{Cb(a/K)}$:= smallest acl^{eq}-closed subset of K with $a \underset{\overline{Cb(a/K)}}{\overset{ACFA}{\downarrow}} K$

independence & CB in ACF $\rightarrow$ $= \bigcup_{b \subseteq_{fin} (\bar{a}, \sigma(\bar{a}), \dots)} acl_\sigma($ generators of the field of definition of the locus $Loc(b/K)$ of the tuple b).

σ-modules

(K,σ) difference field, V finite-dimensional K-vector space

σ-module – (V,Σ), where $\Sigma : (V,+) \xrightarrow{\cong} (V,+)$ such that

$$\Sigma(cv) = \sigma(c)\Sigma(v).$$

Lemma 4.2.

$(V,\Sigma)^{\#} = \{v \in V \mid \Sigma(v) = v\}$ is a $\mathrm{Fix}(K)$-vector space of dimension at most $\dim_K(V)$ & if $(K,\sigma) \models ACFA$, there are $v_1, \ldots v_s \in (V,\Sigma)^{\#}$ such that $V = \langle v_1, \ldots v_s \rangle_K$.

PROOF ...

Lemma 4.2.

$(V,\Sigma)^{\#} = \{v \in V \mid \Sigma(v) = v\}$ is a $\mathrm{Fix}(K)$-vector space of dimension at most $\dim_K(V)$ & if $(K,\sigma) \models ACFA$, there are $v_1, \dots v_s \in (V,\Sigma)^{\#}$ such that $V = \langle v_1, \dots v_s \rangle_K$.

PROOF. (of $\boxed{}$)

Choose a basis B of V over K and let A be the matrix such that, if $v = \sum_{b \in B} a_b b$, then

$$\Sigma(v) := A\,(\sigma(a_b))_{b \in B}.$$

call $\sigma(v) := \sum_{b \in B} \sigma(a_b)\, b$

Then, as σ-modules, $(V,\Sigma) \cong (K^d, A\sigma)$ for some $d \in \mathbb{N}$, so we work with $(K^d, A\sigma)$ instead. In this case, for $v \in K^d$, we have

$$\Sigma(v) = v \iff \sigma(v) = A^{-1} v$$

so we look for $U \in GL_d(K)$ with $\sigma(U) = A^{-1} U$. Such a matrix exists since (K,σ) is existentially closed. The columns of U form the basis we need. $\square$

Jet spaces

K alg. closed of characteristic zero, $X \subseteq K^n$
irreducible affine with ideal $I_X \subseteq K[x_1, \ldots x_n]$

REVIEW from J1

inside of $K[X] = K[x_1,\ldots x_n]/I_X$, we find for any $a \in X$,

$$\mathfrak{m}_{X,a} = \{ f \in K[X] : f(a) = 0 \}$$

(l-1)th jet space

and for any $\ell \geq 2$, the quotient $\mathfrak{m}_{X,a} / \mathfrak{m}^{\ell}_{X,a} =: (j_{\ell-1}(X)_a)^*$

is a finite-dimensional K-vector space.

Explicit description. $X \subseteq K^n$ subvariety, $a \in X$, $m \geq 1$

$$d = \left| \left\{ \frac{\partial^s}{\partial x_{i_1}^{s_1} \cdots \cdots \partial x_{i_r}^{s_r}} : 0 < s \leq m,\ 1 \leq i_1 < \cdots < i_r \leq n,\ s_1 + \ldots + s_r = s,\ s_i > 0\ \forall i \right\} \right| = \mathbb{D}$$

$$\Rightarrow \quad j_m(X)_a \cong V_a = \{ (u_D)_{D \in \mathbb{D}} : \sum_{D \in \mathbb{D}} DP(a)\, u_D = 0,\ \forall P \in I_X \} \subseteq K^d.$$

Important facts: X, W irreducible affine varieties, $W \subseteq X \times X^{\sigma}$ such that $\pi_1 : W \to X$, $\pi_2 : W \to X^{\sigma}$ are dominant and finite-to-one generically. Then if $(a, \sigma(a)) \in W$ is a generic point, for any $m \geq 1$, there is $f_m : j_m(X)_a \xrightarrow{\cong} j_m(X^{\sigma})_{\sigma(a)}$ (whose graph is exactly $j_m(W)_{(a,\sigma(a))}$).

Moreover, $(j_m(X)_a, f^{-1}\sigma)$ is a σ-module over $(\mathcal{U}, \sigma)$ and, for any $K \subseteq K_1$ alg. closed, if X_1 is the variety over K_1 with generic point a, then one can embed $j_m(X_1)_a$ into $(j_m(X)_a, f^{-1}\sigma)$ as σ-modules.

We prove the "moreover".

Moreover, $(j_m(X)_a, f^{-1}\sigma)$ is a σ-module over $(\mathcal{U},\sigma)$ and, for any $K \subseteq K_1$ alg. closed, if X_1 is the variety over K_1 with generic point a, then one can embed $j_m(X_1)_a$ into $(j_m(X)_a, f^{-1}\sigma)$ as σ-modules.

PROOF: note that, if $j_m(X)_a = \{(u_D)_{D\in\mathbb{D}} : \sum_{D\in\mathbb{D}} DP(a)u_D = 0,\ P\in I_X\}$,

$$
\begin{aligned}
j_m(X^\sigma)_{\sigma(a)} &= \{(u_D)_{D\in\mathbb{D}} : \sum_{D\in\mathbb{D}} DP^\sigma(\sigma(a))u_D = 0,\ P\in I_X\} \\
&= \{(\sigma(u_D))_{D\in\mathbb{D}} : \sum_{D\in\mathbb{D}} DP(a)u_D = 0,\ P\in I_X\} \\
&= (j_m(X)_a)^\sigma,
\end{aligned}
$$

hence $f^{-1}\circ\sigma : (j_m(X)_a, +) \xrightarrow{\cong} (j_m(X)_a, +)$ and, since f is a linear iso, we have $(j_m(X)_a, f^{-1}\sigma)$ is a σ-module.

Moreover, $(j_m(X)_a, f^{-1}\sigma)$ is a σ-module over $(\mathcal{U},\sigma)$ and, for any $K \subseteq K_1$ alg. closed, if X_1 is the variety over K_1 with generic point a, then one can embed $j_m(X_1)_a$ into $(j_m(X)_a, f^{-1}\sigma)$ as σ-modules.

PROOF: Let W_1 be the irreducible variety over K_1 whose generic point is $(a,\sigma(a))$. Now, $j_m(W_1)_{(a,\sigma(a))}$ is the graph of an isomorphism

$$f_1 : j_m(X_1)_a \xrightarrow{\cong} j_m(X_1^\sigma)_{\sigma(a)}$$

and then $f_1 = f|_{j_m(X_1)_a}$, which yields the required embedding. $\blacksquare$

PART 2

Zilber's trichotomy

& consequences

Almost internality

$(\mathcal{U}, \sigma) \models ACFA$ monster model

$K \subseteq \mathcal{U}$ alg. closed difference subfield, $p(x) \in S_1(K)$, X 0-definable set

* $p(x)$ is **almost internal** to X if, for some $K \subseteq A$, $a \models p$ with $a \underset{K}{\downarrow} A$, then $a \in acl(A, X)$.

* X is **finite-dimensional** if there is $N \in \mathbb{N}$ such that, for all $a \in X$,

$$\operatorname{trdeg}\big(\underbrace{K\langle a\rangle}_{\text{difference field generated by } K \text{ and } a}/K\big) \leq N.$$

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this is the "canonical base property"!

PROOF : $a \in Y$

$\Downarrow$

WMA $K(a)^{alg} = K(\sigma(a))^{alg}$ $\leftarrow$ replace a by finite $(a, \sigma(a), \dots \sigma^r(a))$

$\Downarrow$

$W \subseteq X \times X^\sigma$ as in the lemma $\rightarrow$

X/K - variety with generic point a

W/K - variety with generic point $(a, \sigma(a))$

$\tilde{X}/K_1$ - variety with generic point a

$\Downarrow$

$c = \overline{Cb(tp(a/K_1))}$ generates smallest field of definition of $\tilde{X}$

So: $a \in Y$ $\rightsquigarrow$ $\overline{Cb(tp(a/K_1))}$ $=: c$ $\rightsquigarrow$ $\tilde{X}$

point weak canonical base algebraic variety

$\Rightarrow$ $\tilde{X}$ is determined by $jm(\tilde{X})_a \subseteq jm(X)_a$ for all $m \in \mathbb{N}$,

by compactness we only need to check $j_M(\tilde{X})_a \subseteq j_M(X)_a$ for $M \gg 0$

$\tilde{X} = V(f_1, \dots f_\ell)$, $f_i = f_i(c)$ $\rightsquigarrow$ we may look at $(\tilde{X}_d)_{d \in \mathcal{U}}$,

$\tilde{X}_d = V\left(f_1(d), \dots f_\ell(d)\right)$

$\tilde{X}_d = \tilde{X} \Leftrightarrow jm(\tilde{X}_d)_a = jm(\tilde{X})_a \;\forall m \in \mathbb{N} \overset{\text{compact.}}{\Longleftrightarrow} jm(\tilde{X}_d)_a = jm(\tilde{X})_a,\ m \gg 0$

$\Rightarrow$ if $f: j_M(\tilde{X})_a \cong j_M(\tilde{X}^\sigma)_{\sigma(a)}$, then we get $j_M(\tilde{X})_a \underbrace{\subseteq}_{\sigma\text{- submodule}} (j_M(X)_a, f^{-1}\sigma)$.

Now, choose a basis $b \subseteq (j_M(X)_a, f^{-1}\sigma)^{\#}$, i.e. rewrite $(j_M(X)_a, f^{-1}\sigma) \cong (K^d, \sigma)$

so that $j_M(\tilde{X})_a \cong L_M$ $\underbrace{\text{defined over } \mathrm{Fix}(\mathcal{U})}$.

↳ $e_M \underset{\text{finite}}{\subseteq} \mathrm{Fix}(\mathcal{U})$, coefficients of equations

$$\Rightarrow \quad c \in K(\underbrace{a}_{\text{point}}, \underbrace{b}_{\text{basis}}, \underbrace{e_M}_{\text{parameters for } L_M}).$$

moral: c depends algebraically on independent data over $\mathrm{Fix}(\mathcal{U})$

$\blacksquare$

A few words on SU-rank: $p \in S(K)$, $a \models p$,

$$\begin{cases} SU(p) \geq 0, \\ SU(p) \geq \alpha,\ \alpha \text{ limit} \iff SU(p) \geq \beta \ \forall \beta < \alpha \\ SU(p) \geq \alpha + 1 \iff \text{there is } K \subseteq F \text{ with} \\ \qquad a \not\perp_K F \ \& \ SU(a/F) \geq \alpha. \end{cases}$$

$\rightsquigarrow$ $SU(p)$ = least ordinal $SU(p) \not\geq \alpha + 1$.

Facts. $SU(a/K) = 1$: $a \notin acl_\sigma(K)$ and for every $K \subseteq F$, either $a \perp_K F$ or $a \in acl_\sigma(F)$. Further, a is transformally algebraic over K.

Setting up for J4:

we will actually need ...

Zilber's trichotomy:

$K \subseteq \mathcal{U} \models ACFA$ algebraically closed,

$p \in S(K)$ of SU-rank 1, then either

p is $\boxed{\text{modular}}$ or $\boxed{\text{almost internal to } Fix(\mathcal{U})}$.

$a \models p^n$, $b \models p^m$, then

$Cb\left(a/K\langle b\rangle\right) \in acl\left(K\langle a\rangle\right)$

there is $b \models p$ &

$K \subseteq A$ with $b \mathop{\downarrow}_K A$ and

$b \in acl(A, Fix(\mathcal{U}))$

PROOF. Suppose p is non-modular: take $a \models p^n$, $b \models p^m$ with $c := Cb(a/K\langle b\rangle) \notin acl(K\langle a\rangle)$. Now by SU-rank 1, $trdeg(K\langle a\rangle/K) < \infty$, so the theorem applies and $tp(c/K\langle a\rangle)$ is almost internal to $Fix(\mathcal{U}) =: k$, i.e. there is $K \subseteq A$ with $c \mathop{\downarrow}_K A$ and $c \in acl(A, k)$.

Rearrange $\bar{b}$ as follows: there is $m_0 \in \mathbb{N}$ such that

$$b_j \in acl(b_1 \cdots b_{m_0}, K) \ \forall j \geq m_0+1 \ \& \ b_i \mathop{\downarrow}_K b_1 \cdots b_{i-1} \ \forall i \leq m_0.$$

Now, by definition $\bar{c} \mathop{\not\downarrow}_K b_1 \cdots b_{m_0}$, hence for some $i \leq m_0 - 1$, $\bar{c} \mathop{\not\downarrow}_{Kb_1 \cdots b_i} b_{i+1}$, i.e. $b_{i+1} \in acl(Kb_1 \cdots b_i \bar{c}) \subseteq acl(Ab_1 \cdots b_i, k)$ & further, we may assume $b \mathop{\downarrow}_{Kc} A$, hence $b \mathop{\downarrow}_K A$ (because $\bar{c} \mathop{\downarrow}_K A$). Thus, $b_{i+1} \mathop{\downarrow}_K Ab_1 \cdots b_i$.

This is exactly showing that p is almost internal to k:

there is $b_{i+1} \models p$, and there is $K \subseteq Ab_1 \cdots b_i$, with

$$b_{i+1} \mathop{\downarrow}_{K} Ab_1 \cdots b_i \quad \& \quad b_{i+1} \in acl(Ab_1 \cdots b_i k).$$

$\square$