Ax-Kochen/Ershov I: The Henselian Menace

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This is the first of two talks on the classical model theory of henselian valued fields. Following Chatzidakis, we prove that an henselian equicharacteristic zero valued field eliminates quantifiers in the Pas language, and derive the Ax-Kochen/Ershov principle.

James Ax (10 January 1937 – 11 June 2006) and **Simon Kochen** (14 August 1934 –) shared the *Frank Nelson Cole Prize for number theory* for their results on Diophantine problems in local fields. Roughly at the same time, on the other side of the Berlin Wall, **Yuri Ershov** (1 May 1940 –) proved similar results.

The source are the course notes *Théorie des Modèles des corps valués*, 2008.

Contents

What is a valued field, again?

These concepts were explained already at some point by somebody else. The point of this section is twofold: fix notation and remind everyone of things which might have been forgotten in the long time that has passed since then (despite the reinforced learning technique of providing pizza together with the material).

Definition 1. *A* valuation *on a field K is the datum of a surjective map $v : K \to \Gamma \cup \{\infty\}$, where $(\Gamma, +, <, 0)$ is an ordered abelian group and*

1. $v(x) = \infty$ *if and only if* $x = 0$,
2. $v(xy) = v(x) + v(y)$,
3. $v(x + y) \geq \min\{v(x), v(y)\}$.

Definition 2 (gadgets)**.** *The* valuation ring *of a valued field (K, v) is the subring $\mathcal{O} = \{x \in K \mid v(x) \geq 0\}$. Its unique* maximal ideal *is denoted by $\mathfrak{m} = \{x \in K \mid v(x) > 0\} \subseteq \mathcal{O}$. The quotient field is called the* residue field $k = \mathcal{O}/\mathfrak{m}$.

Some people denote by vK the value group and Kv the residue field. It is a rather useful notation when more valuations appear on the same field, or some coarsening argument is involved. It is also crucial if your blackboard Ks and ks are not distinguishable.

We denote by $\pi : \mathcal{O} \to k$ the quotient map (also called the *residue map*). It is sometimes possible to extend π "coherently" to the whole of K at the cost of some assumptions on K (if you are quietly whispering to yourself "this must be a saturation thing", you're right) and of losing "control" on what happens on $\mathfrak{m}$. We'll discuss this later on.

Example 3 (your friendly neighbourhood valued field)**.** *The p-adics* $(\mathbb{Q}_p, v_p)$ *have* $\mathbb{Z}$ *as value group and* $\mathbb{F}_p$ *as residue field.*

If there is any risk of confusion, we place subscripts on the gadgets: $\mathcal{O}_K$, $\mathfrak{m}_K$, k_K, Γ_K, and so on.

Remark 4. *Let* $\mathcal{L}_r = \{+, -, \cdot, 0, 1\}$ *be the language of rings. Then one might enlarge* $\mathcal{L}_r$ *to a language for valued fields in various ways, e.g.* $\mathcal{L}_v = \mathcal{L}_r \cup \{\mathcal{O}\}$ *or* $\mathcal{L}_d = \mathcal{L}_r \cup \{\mid\}$*.*

Angular components and where to find them

Definition 5. *An* angular component *is a map* $\mathrm{ac} : K \to k$ *such that*

1. $\mathrm{ac}(0) = 0$,
2. $\mathrm{ac}|_{K^\times} : K^\times \to k^\times$ *is a multiplicative group morphism,*
3. $\mathrm{ac}(x) = \pi(x)$ *for any* $x \in \mathcal{O}^\times$, *in other words* ac *extends the residue map on the units.*

In the case $K = k((t))$ there is an obvious choice of ac, namely

$$\mathrm{ac}(\sum_{n \geq N} a_n t^n) = a_N.$$

One might check that if $N = 0$, i.e. $\sum_{n \geq N} a_n t^n \in \mathcal{O}^\times$, then this is exactly the residue map. One might build a similar map on $\mathbb{Q}_p$ by $\mathrm{ac}(p) := 1$.

The name "angular component" is a bit misleading. In fact, if we think of elements of $k((t))$ as generalized Taylor series, the ac map does not return the *coefficient angulaire* of the function, i.e. the first derivative; it returns, instead, the *leading coefficient* of the series.

Angular components arise from sections of the value group: if $s : \Gamma \to K$ is a section of the valuation, then $\mathrm{ac}(x) := \pi(x/s(x))$ is an angular component map. These sections in turn exist under some assumptions on the ambient structure. In most "natural" examples, one can write down these sections explicitly, and thus obtain explicit angular components; in general, it is a matter of saturation.

Lemma 6. *Every valued field has an elementary extension which admits a section of the valuation.*

Sketch. Starting with a pure subgroup $\Delta \leq \Gamma$, together with a partial section $\Delta \to K$, one might always find an elementary extension where this partial section extends. The result follows from iteration. □

The Ax-Kochen/Ershov principle, or rather: Pas' theorem

Definition 7. *We denote by* $\mathcal{L}_{\mathrm{Pas}}$ *the three-sorted language made up by:*

1. *the language of rings* $\mathcal{L}_r = \{+, -, \cdot, 0, 1\}$ *on the sorts* $\mathbf{K}$ *and* $\mathbf{k}$,
2. *the language of ordered abelian groups* $\mathcal{L}_g = \{+, -, <, 0, \infty\}$,
3. *a symbol for a map* $v : \mathbf{K} \to \mathbf{\Gamma}$ *and a symbol for a map* $\mathrm{ac} : \mathbf{K} \to \mathbf{k}$.

(to the tune of 'Let it be')
When I find myself in times of trouble
Valued fields come to me
Complete first-order theories
A-K-E

Definition 8. *Let T_0 be the $\mathcal{L}_{\mathrm{Pas}}$-theory that prescribes, of models $(K, k, \Gamma \cup \{\infty\}; v, \mathrm{ac})$:*

1. *K is a field, Γ is a ordered abelian group and $v : K \to \Gamma \cup \{\infty\}$ is a valuation,*
2. *(K, v) is henselian,*
3. *the map $\pi : \mathcal{O} \to k$ defined by $\pi(x) = \mathrm{ac}(x)$ if $v(x) = 0$, and $\pi(x) = 0$ otherwise, is a surjective ring morphism with kernel $\mathfrak{m}$,*
4. *k is a field of characteristic 0.*

Under these hypotheses we have that $\mathcal{O}/\mathfrak{m} \cong k$, and we identify them. Moreover, given some fixed field k and ordered abelian group Γ, let T be the theory obtained from T_0 by adjoining the full $\mathcal{L}_r$-theory of k and the full $\mathcal{L}_g$-theory of Γ.

The idea is that this theory should capture the whole model theoretical information that contained in a henselian valued field of equicharacteristic zero; in other words, as an $\mathcal{L}_{\mathrm{Pas}}$-structure (K, v) is little more than the sum of k_K and Γ_K. This will follow from completeness, which we obtain for free from a relative quantifier elimination result.

One might say, *Oh, come on, only a **relative** result?*. As underwhelming as it is, one cannot really hope for more; if k_K and Γ_K are *bad*, then (K, v) ought to be *atleast* as bad: both these structures are interpretable in (K, v). Thus, all results around the model theory of (K, v) will have to be *relative* to the properties of k_K and Γ_K. If your residue field is e.g. $\mathbb{Q}$, you can't really expect the valued field to be a decent, polite valued field.

Theorem 9 (Pas)**.** *T eliminates the* **K***-quantifier.*

This partial form of quantifier elimination will follow from a back-and-forth lemma:

Lemma 10. *Suppose Σ is a set of $\mathcal{L}$-formulae closed under Boolean combinations. Let T be a theory and $\kappa > |\mathcal{L}|$. Then the following is sufficient to obtain quantifier elimination down to formulae in Σ: given two κ-saturated models M, N of T and an isomorphism between substructures $f : A \subseteq M \to B \subseteq N$ with $|A| < \kappa$ that preserves Σ-formulae, for any $a \in M$ there is an isomorphism $g \supseteq f$ between substructures of M and N that preserves Σ-formulae and whose domain contains a.*

We then set-up a back-and-forth of this form: start with two $\mathcal{L}_{\mathrm{Pas}}$-structures $(K, \Gamma_K, k_K), (L, \Gamma_L, k_L) \models T$ which are $\aleph_1$-saturated. Let Σ be the set of formulae which only contain quantifiers over $\mathbf{\Gamma}$ and $\mathbf{k}$. Choose (A, Γ_A, k_A) and (B, Γ_B, k_B) countable substructures respectively of M and N. Given an isomorphism $f : A \to B$ that preserves Σ-formulae, and $a \in M \setminus A$, we wish to extend f to a isomorphism of substructures whose domain contains a. To do so, we extend f to some $(C, \Gamma_C, k_C) \preceq (K, \Gamma_K, k_K)$, countable, with $a \in C$; the procedure requires interweaving several steps ("dovetailing"):

Note that the isomorphism $f : A \to B$ is an isomorphism of substructures, hence it is really threefold: there is an isomorphism $f : A \xrightarrow{\sim} B$ of valued fields, an isomorphism $f_r : k_A \xrightarrow{\sim} k_B$ of fields, and an isomorphism $f_v : \Gamma_A \xrightarrow{\sim} \Gamma_B$. The latter two can really be recovered from the first one, by composing with the relevant maps.

1. extend k_A to k_C, obtaining (A, Γ_A, k_C),

2. extend Γ_A to Γ_C, obtaining (A, Γ_C, k_C),
3. replacing A with A^h,
4. extending A to make π surjective on k_C,
5. extending A to make v surjective on Γ_C,
6. extending (A, Γ_C, k_C) to the immediate extension (C, Γ_C, k_C).

This will be done in one of the next sections. But before that, let us see an application.

The Ax-Kochen/Ershov principle, or rather: Pas' corollary

Corollary 11. *Given (K, v) and (L, w) henselian valued fields of equicharacteristic 0. Then,*

$$(K, v) \equiv (L, w) \iff [k_K \equiv_{\mathcal{L}_r} k_L \wedge \Gamma_K \equiv_{\mathcal{L}_g} \Gamma_L].$$

The $\equiv$ on the LHS is left purposefully generic. This is true for any language of valued fields, even without an angular component: this can be eliminated upon moving to a saturated extension, and is hence not taking any part in determining the theory of the structure.

Proof. Note that we can always move to elementary extensions of (K, v) and (L, w) enriched with ac-maps, hence for the right to left direction it is enough to show that the theory T is complete. To do so, it is enough to notice that two of its models, say (K, Γ_K, k_K) and (L, Γ_L, k_L), both share a substructure, namely $(\mathbb{Q}, \{0\}, \mathbb{Q})$. Since T eliminates the $\mathbf{K}$-quantifier, and $\Gamma_K \equiv \Gamma_L$ and $k_K \equiv k_L$, this implies that $(K, \Gamma_K, k_K) \equiv (L, \Gamma_L, k_L)$.

The left to right direction requires less machinery: both the residue field and the value group are uniformly interpretable in (K, v) and (L, w) (with whatever valued fields language you choose), and hence they are elementarily equivalent if the ambient structures are. □

The actual Ax-Kochen/Ershov principle!

For a family of $\mathcal{L}$-structures $(M_q)_{q \in \mathbb{P}}$, consider a non-principal ultrafilter μ on the set of primes $\mathbb{P}$. Let $\prod_{q \in \mathbb{P}} M_q / \mu$ be the ultraproduct with respect to μ.

Corollary 12. $\prod_{q \in \mathbb{P}} \mathbb{Q}_q / \mu \equiv \prod_{q \in \mathbb{P}} \mathbb{F}_q((t)) / \mu$.

Proof. The two are henselian equicharacteristic 0 valued fields with the same residue field (the ultraproduct $\prod_{q \in \mathbb{P}} \mathbb{F}_q / \mu$) and the same value group ($\hat{\mathbb{Z}}$). □

Before going on to the (rather tiresome) proof, let me mention one more result that might motivate all of this work. As a direct application of these results,

Corollary 13. *Consider finitely many polynomials $f_1, \ldots f_\ell \in \mathbb{Z}[t]$. For all but finitely many primes q, every solution of $f_1(t) = \cdots = f_\ell(t) = 0$ in $\mathbb{F}_p$ gives rise to a solution in $\mathbb{Z}_p$.*

The path to hell is paved with back-and-forth arguments

We finally come to the proof. As stated before, we start with two $\mathcal{L}_{\text{Pas}}$-structures $(K, \Gamma_K, k_K), (L, \Gamma_L, k_L) \models T$ which are $\aleph_1$-saturated. We take Σ to be the set of formulae which only contain quantifiers over $\mathbf{\Gamma}$ and $\mathbf{k}$. We'd like to write down a recipe that, given the following ingredients:

1. (A, Γ_A, k_A) and (B, Γ_B, k_B) countable substructures, respectively of M and N,
2. an isomorphism $f : A \to B$ that preserves Σ-formulae,
3. $a \in M \setminus A$,

produces a new isomorphism of substructures $g \supseteq f$ that still preserves Σ-formulae and whose domain contains a. More precisely, given $(C, \Gamma_C, k_C) \preceq (K, \Gamma_K, k_K)$ which is countable and contains a we wish to extend f to C.

If I forget (I *will* forget) assume that the new isomorphism is expected to respect Σ-formulae.

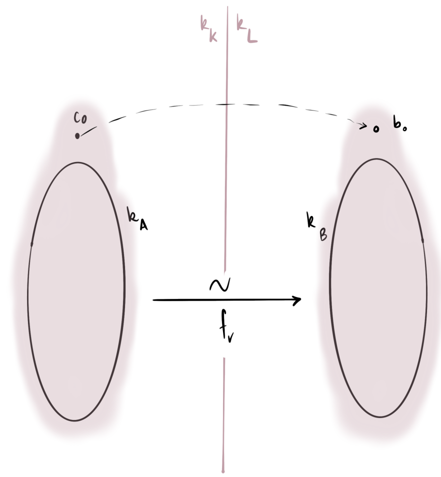
Step 0 (the one that everyone forgets). The language we have chosen for the $\mathbf{\Gamma}$ sort forces $\Gamma_A \leq \Gamma_K$. However, the language we have chosen for $\mathbf{K}$ and $\mathbf{k}$ allows the possibility that A and k_A are only rings – we didn't include the inverse map! This is not really a problem: there is a canonical way to extend A and k_A to a field – namely, moving to the field of fractions. The valuation extends in a canonical way as well: $v(a/b) = v(a) - v(b)$ for any $a, b \neq 0$ from K; further, $\text{ac}(a/b) = \text{ac}(a)/\text{ac}(b)$. The isomorphism f extends uniquely to an isomorphism of the fraction fields (again, $f(a/b) = f(a)/f(b)$ for $a, b \neq 0$).

It is then safe to assume that both A and k_A are fields.

Step 1: extending k_A to k_C.

By saying "extending k_A to k_C," what I really mean is: prescribe a unique recipe to extend the given map f to the structure (A, Γ_A, k_C), so that we may assume that $k_A = k_C$. This wording will appear again in the next steps, always with this meaning. In fact, the whole proof rests on the "assumption" – read, Steps 1-5 – that one can only work with immediate extensions.

Since k_C is countable, let $(c_i)_{i<\omega}$ be an enumeration of it. Before writing down the details, who might obscure the content, let me explain the idea: a "new" element – say, c_0 – that has to be added to k_A must be mapped to an element $b_0 \in k_L$ that has the same relationship to k_B as c_0 has to k_A. This is encoded in the $\mathcal{L}_r(k_A)$-type of c_0, which can be readily translated into an $\mathcal{L}_r(k_B)$-type by letting f_r – which is an isomorphism, so "preserves information" in the strongest sense possible – act on the parameters. Formally, consider the type $p(x) = \text{tp}_{\mathbf{k}}(c_0/k_A)$. Consider the type $q(y)$ given by the following procedure: if $\varphi(x, \bar{c}) \in p(x)$, where $\bar{c}$ is a tuple from k_A, then $\varphi(y, f_r(\bar{c})) \in q(y)$.



Since L is $\aleph_1$-saturated, we can realize $q(y)$: let $b_0 \vDash q(y)$. Then the map $c_0 \mapsto b_0$ extends to an isomorphism $f'_r : k_A(c_0) \to k_B(b_0)$.

Note that $p(x)$ records whether c_0 was algebraic or transcendental over k_A, and hence the same will hold for b_0 over k_B. The construction of the new isomorphism is then a purely field-theoretic question.

We can thus extend f to the isomorphism of $\mathcal{L}_{\mathrm{Pas}}$-structures

$$(f, f_v, f'_r) : (k_A, \Gamma_A, k_A(c_0)) \xrightarrow{\sim} (k_B, \Gamma_B, k_B(b_0)).$$

We need to check that the new isomorphism preserves Σ-formulae. However, a moment of unpleasant yet straightforward syntactic reflection will bring us to the conclusion that, to preserve Σ-formulae, one really only needs to preserve formulae of the form

$$\psi_0(x_0) \wedge \psi_1(v(t_1(x)), y_1) \wedge \psi_2(\mathrm{ac}(t_2(x)), y_2)$$

where $\psi_0(x_0)$ is a *quantifier-free* $\mathcal{L}_r$-formula with x_0 of sort $\mathbf{K}$, $\psi_1(x_1, y_1)$ is an $\mathcal{L}_g$-formula and $\psi_2(x_2, y_2)$ is an $\mathcal{L}_r$-formula, with x_2, y_2 of sort $\mathbf{k}$, and further t_1 and t_2 are tuples of terms obtained from the field operations. After this shortcut, it is relatively immediate to notice that the new isomorphism preserves these formulae (essentially because of the compatibility with ac and v).

As the $\mathbf{K}$-sort of A didn't change, we don't need to check what happens to v and ac, as the compatibility will be automatically satisfied.

We repeat the procedure we have just described countably many times, thus exhausting $(c_i)_{i<\omega}$. We can hence assume that f is an isomorphism defined on the substructure structure (A, Γ_A, k_C).

Step 2: extending Γ_A to Γ_C.

This procedure mirrors the procedure in Step 1: we take a new element in Γ_C, take its type over Γ_A, realize it on the other side of the river by virtue of the isomorphism f_v and map one new element to the other. The new isomorphism will then again preserve Σ-formulae. Note that this procedure didn't touch the $\mathbf{k}$-sort, hence we can repeat it countably many times without breaking what we did in Step 1. From now on, we can assume that f is defined on (A, Γ_C, k_C).

Interlude: in the next steps, we shall seek to extend f on the $\mathbf{K}$-sort, while still preserving Σ-formulae. Note that, by the annoying syntactic meditation we performed a couple of paragraphs above, this is unnecessary: if we extend f to f', defined on some A', then surely f' will preserve ψ_0 (as it is quantifier-free), f'_v will preserve ψ_1 (since for any $a \in A'$ we have $v(t_1(a)) \in v(A) = \Gamma_C$) and f'_r will preserve ψ_2 (since for any $a \in A'$ we have $\mathrm{ac}(t_2(a)) \in k_A = k_C$). By virtue of Steps 1 and 2, we then only need to check that f' is a $\mathcal{L}_{\mathrm{Pas}}$-isomorphism.

Step 3: extending A to A^h.

Note that C (as a valued field) is henselian, since henselianity is a first-order property. In particular, then, by the universal property of henselizations C will contain a copy (over A) of A^h. Let's call it A^h. One can then argue that $A^h = C \cap A^{\mathrm{alg}}$, and similarly $B^h = L \cap B^{\mathrm{alg}}$. In particular, f extends to $f' : A^h \xrightarrow{\sim} B^h$. We need to check that is a $\mathcal{L}_{\mathrm{Pas}}$-isomorphism: as A^h is an immediate extension (i.e., it has the same residue field and value group), given $a \in A^h \setminus A$, there is $a' \in A$ such that $v(a - a') > v(a) = v(a')$, in particular then $a = a'(1 + u)$ with $v(u) > 0$, so $\mathrm{ac}(a) = \mathrm{ac}(a')$. Hence $v^h(f(a) - f(a')) > v^h(f(a))$, and $\mathrm{ac}(f(a)) = \mathrm{ac}(f(a')) = f(\mathrm{ac}(a))$.

The new map (f', f_v, f_r) defined on (A^h, Γ_C, k_C) is a isomorphism of $\mathcal{L}_{\mathrm{Pas}}$-structures, and thus we may assume that $A = A^h$ (i.e., A is henselian).

Interlude: at this point, it is worth noting that it is entirely possible that $v(A^\times) \subsetneq \Gamma_A$ and $\mathrm{ac}(A) \subsetneq k_A$. In other words, the residue field of A might be strictly smaller than k_A, and its value group might be strictly smaller than Γ_A. We then have to extend f to something with residue field k_A or, from the opposite point of view, "lift" the full k_A to an extension of A. This step will take the difficulty of the argument up a notch.

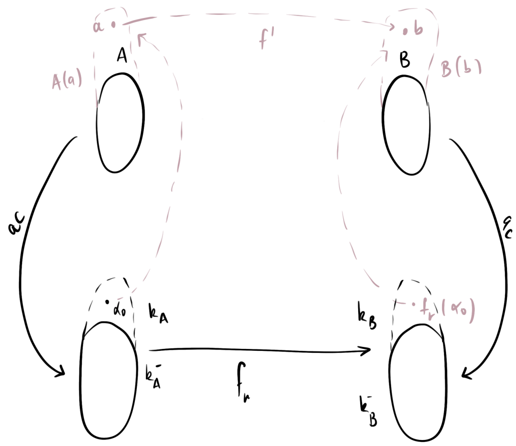
Step 4: extend f to a subfield $D \subseteq C$ such that $\pi(\mathcal{O}_D) = k_D$.

Denote by k_A^- the residue field of A, with valuation ring $\mathcal{O}_A$. Similarly, denote by k_B^- the residue field of B, with valuation ring $\mathcal{O}_B$. Take $\alpha_0 \in k_C \setminus k_C^-$. We have two possibilities: either α_0 is algebraic over k_A^-, or it is trascendental.

Step 4.a: α_0 is algebraic over k_A^-. Let $P(t) \in \mathcal{O}_A[t]$ be such that $\bar{P}(t)$ is the minimal polynomial of α_0 over k_A^-, and P and $\bar{P}$ have the same degree. Then $P(t)$ is also irreducible and, since k_K has characteristic zero, α_0 is a simple zero of $\bar{P}$ and hence it lifts to $a \in \mathcal{O}_C$ by henselianity. In particular, $P(a) = 0$ and $\pi(a) = \alpha_0$.

Let $P^f(t)$ be the polynomial obtained by letting f act on the coefficients of P. As f is an isomorphism, $\bar{P}^{f_r}(t)$ is irreducible over k_B^- and has $f_r(\alpha_0)$ as a simple root. By henselianity once again, there is $b \in L$

such that $\pi(b) = f_r(\alpha_0)$ and $P^f(b) = 0$. We extend f to f' defined on $A(a)$ in the only way imaginable: by extending the map $a \mapsto b$.



Note that, if $n = \deg P$, then $1, \pi(a), \ldots \pi(a^{n-1})$ are k_A^--linearly independent, and hence

$$v(\sum_{i=0}^{n-1} c_i a^i) = \min_{i=0,\ldots n-1} \{v(c_i)\}.$$

Similarly, $1, \pi(b), \ldots \pi(b^{n-1})$ are k_B^--linearly independent. In particular, the map f' is an isomorphism of valued fields between $A(a)$ and $B(b)$. As the value groups of A and $A(a)$ are the same, any element of $A(a)$ can be written as ub, where $v(u) = 0$ and $b \in A$. Hence f' commutes both with π and ac, in particular it is a $\mathcal{L}_{\text{Pas}}$-isomorphism.

Step 4.b: α_0 is not algebraic over k_A^-. Pick $a \in C$ with $\pi(a) = \alpha_0$ and $b \in L$ with $\pi(b) = f_r(\alpha_0)$. Both a and b are transcendental over A and B and hence, for any $c_0, \ldots c_n \in A$,

$$v(\sum_i c_i a^i) = \min_i v(c_i),$$
$$v'(\sum_i f(c_i) b^i) = \min_i v'(f(c_i)) = f_v(\min_i v(c_i)) = f_v(v(\sum_i c_i a^i)),$$

so the map $f' : A(a) \to B(b)$ is an isomorphism of valued fields, which is again a $\mathcal{L}_{\text{Pas}}$-isomorphism.

Upon repeating this procedure, we may assume that $k_A^- = k_A = k_C$, in other words that the residue field of A is precisely k_C.

Once again, it is worth reminding that by "f' is a $\mathcal{L}_{\text{Pas}}$-isomorphism" I really mean that (f', f_r, f_v) is a $\mathcal{L}_{\text{Pas}}$-isomorphism.

Step 5: extend f to a subfield E of C such that $v(E^\times) = \Gamma_C$.

As before, denote by Γ_A^- the value group of A. This procedure will strongly imitate the previous Step: the dichotomy algebraic *vs.*

transcendental will be substituted by the dichotomy torsion modulo Γ_A *vs.* no torsion modulo Γ_A^-.

Suppose $\alpha \in \Gamma_A \setminus \Gamma_A^-$, with $\alpha > 0$.

Step 5.a: assume that for all natural numbers $n > 0$ we have $n\alpha \notin \Gamma_A^-$. Then, given $a \in C$ with $v(a) = \alpha$, we necessarily have that a is transcendental over A: otherwise, if for example

$$a^n + c_{n-1}a^{n-1} + \cdots + c_0 = 0$$

for $c_0, \ldots c_{n-1} \in A$, then

$$n\alpha = v(a^n) = v(\sum_{i=0}^{n-1} -c_{n-1}a^i) = \min_{i=0,\ldots n-1} v(c_i) \in \Gamma_A^-.$$

Similarly, if we choose $b \in L$ such that $v'(b) = f_r(\alpha)$, then b is transcendental over B. Without loss of generality, we may assume that $\mathrm{ac}(a) = 1$ and $\mathrm{ac}'(b) = 1$, and then the extension f' of f to $A(a)$, mapping a to b, is a valued field isomorphism, and hence (f', f_r, f_v) is an $\mathcal{L}_{\mathrm{Pas}}$-isomorphism.

Step 5.b: suppose there is $n > 0$ such that $n\alpha \in \Gamma_A^-$. Take N such minimal. Then we can choose $a \in C$ with $v(a) = \alpha$ and $a^N \in A$, and similarly $c \in L$ with $c^N \in B$ and $v'(c) = f_v(\alpha)$. The element c is thus algebraic over B, and since $f(k_C) \preceq k_L$, we can find $d \in \mathcal{O}_B$ such that

$$\pi(d) = f_r(\mathrm{ac}(a))\mathrm{ac}(c^{-1})$$

and thus, modulo multiplying c by d, we can assume without loss of generality that $\mathrm{ac}(c) = f_r(\mathrm{ac}(a))$. Then

$$f(a^N) = c^N(1 + u)$$

with $v'(u) > 0$. We may choose $d \in L$ with $\pi(d) = 1$ and $d^N = 1 + u$, so let $b := cd$. By construction, $f' : A(a) \to B(b)$ sending a to b is an $\mathcal{L}_{\mathrm{Pas}}$-isomorphism.

By iterating this procedure, we may assume that $\Gamma_A^- = \Gamma_A = \Gamma_C$.

Now, a moment of reflection: the Steps 3, 4, and 5 have not touched Γ_A or k_A in any way. All of the modifications happened on the **K**-level, on A, and thus we haven't ruined all of our previous work. Phew!

Step 6: the final rush. You might think to yourself: this is easy! As we did in Step 1 and 2, we take a new element $\alpha \in C \setminus A$, identify its type over A, translate it to the other side and realize it to obtain a potential image of α. There is a crucial subtlety here. We will, in a moment, realize that new elements can only be transcendental over A; however, A has more structure that k_A: there is a valuation, which gives us a further layer, or *dimension*, to think about. While any polynomial $P(t) \in A[t]$ will not vanish on α, by virtue of α being transcendental over A, it might very well be that P tries *very hard* to vanish on α. In this case, *very hard* means something like *at the limit* – for example, it might very well be that $v_C(P(t))$ becomes bigger and bigger as we approach α. In this case, we'd say that α is of *algebraic type*; otherwise, we'd say α is of *transcendental type*. If we, somehow naively, only took some $\beta \in L$ with the same type over B, then it might very well be that this limit behaviour is not preserved by the natural map $A(\alpha) \to B(\beta)$ (note that β will be transcendental over B). We might, in other words, inadvertedly pick β of transcendental type.

Crucially, we'll use the following two results:

1. Any henselian valued field K in equicharacteristic zero is algebraically maximal, i.e. it admits no immediate extension of algebraic type,
2. Any two immediate extensions of transcendental type are canonically isomorphic over K.

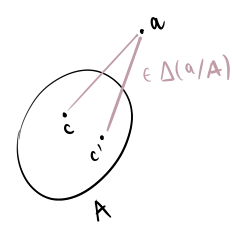
Luckily for us, *Kaplansky theory* tells us that not only henselian valued fields in equicharacteristic zero do not have algebraic extensions, they also don't admit extensions of *algebraic type*; in other words, α will be necessarily of transcendental type, and β as well, so that $A(\alpha) \to B(\beta)$ is really an isomorphism of valued fields.

First of all, let us notice that no immediate algebraic extension is available: by Ostrowski's formula, such an extension $K \subset L$ would satisfy

$$[L : K] = [k_L : k_K][\Gamma_L : \Gamma_K],$$

from which we deduce $[L : K] = 1$. Hence, any choice of α is transcendental. Now, α is also of transcendental type, hence for any polynomial $P \in A[t]$ there is $\delta \in \Delta(\alpha/A) = \{v_C(\alpha - c) \mid c \in A\}$ such that $v_C(P(t))$ is constant on $B_\delta(\alpha) \cap A$. By saturation, we might choose $\beta \in L$ such that

$$v_L(\beta - f(c)) = f_v(v_C(\alpha - c)),\ \forall c \in A.$$



Then $v_L(P^f(t))$ will be constant on $B_{f_v(\delta)}(f(\alpha'))$ for any choice of $\alpha' \in B_\delta(\alpha) \cap A$. In particular, β will also be of transcendental type over B, and hence there is a unique isomorphism of valued fields $f' : A(\alpha) \to B(\beta)$ that extends f. Thus (f', f_r, f_v) is an $\mathcal{L}_{\text{Pas}}$-isomorphism as requested.

We can then use Step 3 to extend f' to $A(a)^h = A(a)^{\text{alg}} \cap C$. Upon iterating this procedure countably many times, we have finally achieved $A = C$. We can repeat all of the above steps on the other side of the river, for B, and we conclude the proof.

Sources

1. Zoé Chatzidakis, *Théorie des Modèles des corps valués*
2. Martin Hils, *Model Theory of Valued Fields*
3. Lou van den Dries, *Lectures on the model theory of valued fields*